Asymptotics of reproducing kernels and recurrence coefficients

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Contents

Preface

0.1 Introduction and statement of results

In this thesis we discuss results related to orthogonal polynomials, specifically results regarding recurrence coefficients and reproducing kernels. We define orthogonal polynomials p_n with $\deg p_n = n$ through a Borel measure μ with support $\Omega \subset \mathbb{R}$ and the property that

$$\int_\Omega p_i(x)p_j(x)d\mu(x) = 0 \text{ if } i \neq j$$

$$\int_\Omega p_i(x)p_j(x)d\mu(x) \neq 0 \text{ if } i = j$$

Furthermore, we will restrict ourselves to the case that we can write $d\mu(x) = w(x)dx$ and we will refer to $w(x)$ as the *weight function* and Ω is throughout this thesis typically either $[-1, 1]$ or $\mathbb{R}$.
We say that p_n is *orthonormal* if

$$\int_\Omega p_i(x)p_j(x)d\mu(x) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Orthonormal polynomials will be represented by p_n.
We say that a polynomial is *monic* if its leading coefficient is equal to 1. Monic orthogonal polynomials will be represented by π_n.
By recurrence coefficients a_n, b_n we mean the coefficients found in the three term recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x) \tag{0.1.1}$$

(see [64]).
By a reproducing kernel, or rather a *normalised* reproducing kernel, we mean a function

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}}$$

If our orthogonality is defined through a varying weight, which means a weight dependent on a parameter N, we will symbolise this by adding a subscript N to any mathematical object involved. So w becomes w_N, p_n becomes $p_{n,N}$, π_n becomes $\pi_{n,N}$, a_n becomes $a_{n,N}$, b_n becomes $b_{n,N}$ and K_n becomes $K_{n,N}$.

The theory of orthogonal polynomials has a multitude of applications to a variety of fields ranging from statistical physics (see for example [8]), to aerodynamics (see for example [7]) to random matrices (see for example [14], [15], [19] and many more) to discrete equations (see for example [12]).

Techniques that give information about general behaviour of orthogonal polynomials are therefore of paramount importance.

The Deift-Zhou method of steepest descent provides in this need (see for example [19], [43] and chapter 2) and will be our main tool for deducing asymptotic behaviour of recurrence coefficients and reproducing kernels.

The Deift-Zhou method of steepest descent is a method that enables us to obtain asymptotics of orthogonal polynomials through performing elementary yet ingenious operations on a related Riemann-Hilbert problem.

Roughly speaking, whenever polynomials $\{p_n(x)\}_{n=0}^{\infty}$ are orthogonal with respect to an analytic weight w, Deift-Zhou steepest descent analysis allows us to deduce asymptotics for polynomials and related objects, such as recurrence coefficients and reproducing kernels (see for example chapter 2 and chapter 4).

It is our aim in this thesis to further explore this method and refine some of its results as described in the following outline:

0.1.1 Chapter 1: Riemann-Hilbert problems

In chapter 1 we will introduce the concept of a Riemann-Hilbert problem through a series of examples and discuss the main Riemann-Hilbert problems in this thesis. The Riemann-Hilbert problem for orthogonal polynomials is taken from [33]. Its discussion is based on results from [19], [27] and [43]. The Riemann-Hilbert problems for the 1×1 case are based on [64], the other 2×2 Riemann-Hilbert problems are based on results from [19], [32], [40] and [43].

Chapter 1 serves as an overview of Riemann-Hilbert problems and does not contain new results or techniques.

0.1.2 Chapter 2: The Deift-Zhou steepest descent analysis

In chapter 2, we will give an example of Deift-Zhou steepest descent analysis, based on the Riemann-Hilbert problem for orthogonal polynomials as described in chapter 1. The analysis is strongly based on [32], [33] and [43] and is mainly necesary to obtain limit behaviour for the reproducing kernel, which is needed in chapter 5. As such, chapter 2 does not contain new results or techniques.

0.1.3 Chapter 3: Relations between limiting kernels

In chapter 3 we will show how Deift-Zhou steepest descent analysis can be used to relate different forms of limit behaviour of reproducing kernels to each other. The technique is based on [26], [41] and [42].
Specifically, we study the following kernels:

- The *sine kernel*

$$\frac{\sin(\pi(x-y))}{\pi(x-y)}$$

- The *Bessel kernel*

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}$$

  where J_α is the Bessel function (see section A.3.1).

- A second Bessel kernel

$$\mathbb{J}^0_\alpha(x,y) = \pi\left(\frac{|x|}{x}\right)^\alpha\left(\frac{|y|}{y}\right)^\alpha\sqrt{x}\sqrt{y}\frac{J_{\alpha+\frac{1}{2}}(\pi x)J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x)J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)}$$

  where $\alpha > -\frac{1}{2}$ and $J_{\alpha\pm\frac{1}{2}}$ is the Bessel function of order $\alpha\pm\frac{1}{2}$ (see [3], [34] and Remark 1.2 of [47]). Also, all functions used for $\mathbb{J}^0_\alpha$ have cuts along the negative real line (where applicable). For negative values of x, we will write $x^\alpha = e^{\alpha\pi i}|x|^\alpha$ and $\sqrt{x} = e^{\frac{1}{2}\pi i}\sqrt{|x|}$.

- The *Confluent Hypergeometric kernel*

$$\mathbb{K}^{CHF}_c(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda;2\pi ix);G(\lambda;2\pi iy)]$$

  where $\lambda = \frac{i\log c}{\pi}$, $G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$, with $\phi(a,c;z)$ as in (A.6.1) and $[f(x);g(y)] = f(x)g(y) - f(y)g(x)$.

We will prove that

Theorem 0.1.1. For $s > 0$, for all $x, y \in \mathbb{R}$ and $\alpha > -1$,

$$2\pi s\mathbb{J}_\alpha\left(s^2+2\pi xs, s^2+2\pi ys\right) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s\to\infty$.

Theorem 0.1.2. For $s \in \mathbb{R}$, for all $x, y \in \mathbb{R}$ and $c > 0$,

$$\mathbb{K}_c^{CHF}(x+s, y+s) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s \to \pm\infty$.

Theorem 0.1.3. For $s \in \mathbb{R}$, for all $x, y \in \mathbb{R}$,

$$\mathbb{J}_\alpha^0(s+x, s+y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s \to \pm\infty$.

It should be emphasised that while the results of chapter 3 are new, they can be obtained through easier and more straightforward techniques. The relevance of chapter 3 however, lies in the method used, which is based on Deift-Zhou steepest descent analysis and could also be applied to other kernels that do not have other explicit formulas.

0.1.4 Chapter 4: The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights

In chapter 4 we will consider the asymptotic behavior of the recurrence coefficients $a_{n,N}$ and $b_{n,N}$ in the three-term recurrence relation

$$x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$$

for monic orthogonal polynomials with respect to varying exponential weights

$$w_N(x) = e^{-NV(x)}$$

We will prove that

Theorem 0.1.4. Let V be real analytic and one-cut regular. Then there exist constants α_{2m} and β_m, $m = 1, 2, \dots$ (depending on V) such that $a_{n,n}$ and $b_{n,n}$ have the following asymptotic expansions as $n \to \infty$:

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \qquad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m}.$$

The first coefficient β_1 in the expansion for $b_{n,n}$ is given explicitly by

$$\beta_1 = \frac{1}{2\pi(b-a)} \left(\frac{1}{h(b)} - \frac{1}{h(a)} \right)$$

where h is the function appearing in the expression (4.1.1) for the equilibrium measure associated with V.

See chapter 4 for further details.
Chapter 4 corresponds to the published paper [45].

0.1.5 Chapter 5: Limit behaviour of reproducing kernels with respect to a non-analytic weight

In chapter 5 we will introduce a method based on [54] and [53] to generalise results on limit behaviour for reproducing kernels for analytic weights to discontinous weights. For Deift-Zhou steepest descent analysis, this means that it becomes possible to deal with non-analytic weights as well. A central role in this chapter will be played by the following two types of weights:

- Let $\alpha > -1$, $\beta > -1$, $x_0 \in (-1,1)$ and

$$\nu_{x_0}(z) = \begin{cases} c^2 \text{ for Re } z \geq x_0 \\ 1 \text{ for Re } z < x_0 \end{cases} \tag{0.1.2}$$

  with $c > 0$ and $c \neq 1$. Furthermore, let

$$w^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta \text{ where } \alpha > -1,\ \beta > -1 \tag{0.1.3}$$

  and let $h(x)$ be a positive real analytic function for $x \in [-1,1]$. The first type of weight function will be of the form

$$w(x) = h(x) w^{\alpha,\beta}(x) \nu_{x_0}(x)$$

  where $x \in [-1,1]$.

- The second type of weight function will be of the form

$$w_N(x) = H(x) e^{-NV(x)}$$

  Here, V is a real analytic function which has the property that

$$\frac{V(x)}{\log(1+x^2)} \to +\infty \tag{0.1.4}$$

  where $x \in \mathbb{R}$. $H(x)$ is an analytic function that is positive on $\operatorname{supp} \psi_V$ (see [15] or chapter 5 for a definition of ψ_V). We will assume that $\operatorname{supp} \psi_V$ consists of an interval $[a,b]$.

We will illustrate our method by proving the following theorems:

Theorem 0.1.5. Let c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive finite Borel measure on $(-1,1)$ through $d\mu(x) = h(x) w^{\alpha,\beta}(x) dx$, where $c_1 \leq h(x) \leq c_2$ for $x \in [-1,1]$ and continuous on an open subinterval $I \subset (-1,1)$.
Let $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$. For $x \in I$ and u, v in compact subsets of $\mathbb{R}$, we have that uniformly

$$\lim_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}.$$

Theorem 0.1.6. Let $\delta > 0$ and c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive Borel measure on $(-1,1)$ through $d\mu(x) = h(x)w^{(\alpha,\beta)}(x)dx$, $\alpha, \beta > -1$, where

$$c_1 \leq h(x) \leq c_2$$

for $x \in [-1,1]$ and $h(x)$ is continuous for $x \in [1-2\delta, 1] \subset [-1,1]$. Then for u, v in compact subsets of $(0,\infty)$, we have that uniformly

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n \left(1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}\right) = \mathbb{J}_\alpha(u,v)$$

Theorem 0.1.7. Let $I \subset [-1,1]$ be an open interval, $x_0 \in I$ and c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive Borel measure on $(-1,1)$ through

$$d\mu(x) = w(x)dx = h(x)\nu_{x_0}(x)w^{(\alpha,\beta)}(x)dx$$

where $\alpha, \beta > -1$, $x_0 \in (-1,1)$, h is continuous on I and

$$c_1 \leq h(x) \leq c_2$$

for $x \in [-1,1]$. Let ν_{x_0} as in (7.4.1) and $w^{(\alpha,\beta)}$ as in (7.4.2). Let u, v lie in compact subsets of $\mathbb{R}$ and let $x \in I$. We then have that uniformly

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,v)$$

where $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$.

Theorem 0.1.8. Let $K_{n,N}(x,y)$ be the normalised reproducing kernel with respect to a weight function $w_N(x) = H(x)e^{-NV(x)}$, where H is a positive, continuous function, V is real analytic. Then

(a) For $\psi_V(x) > 0$ and for u, v in compact subsets of $\mathbb{R}$ we have that uniformly

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n} \left(x + \frac{u}{n\psi_V(x)}, x + \frac{v}{n\psi_V(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}$$

(b) For b a right edge point of $\{x : \psi_V > 0\}$ and u, v in compact subsets of $\mathbb{R}$ we have that uniformly

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n} \left(b + \frac{u}{(cn)^{2/3}}, b + \frac{v}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

where Ai is the Airy function (see section 1.3.2).

Theorem 0.1.5 and Theorem 0.1.6 were already proven in a more general setting by Lubinsky in [54] and [53] respectively. Part (a) of Theorem 0.1.8 was proven for a more general case by Levin and Lubinsky in [51]. Theorem 0.1.7 and part (b) of Theorem 0.1.8 are new results.

Chapter 1

Riemann-Hilbert Problems

1.1 Introduction

The aim of this chapter is to introduce the concept of a Riemann-Hilbert problem and review the Riemann-Hilbert problems used in this thesis. For a proper historical overview on Riemann-Hilbert problems, please see [39]. Riemann-Hilbert problems are all about representing a piecewise analytic function in terms of its jump behaviour and asymptotics. And, with Liouville's theorem in mind, that's a pretty useful way of representing a function.

Example 1.1.1. We all know that a function $f(z) = z^a$, $a \notin \mathbb{Z}$, is not analytic in the entire complex plane: A ray has to be chosen along which f makes a certain jump. In this case, we will choose this ray to be the positive half line: Let $x \in \mathbb{R}_{\geq 0}$. Then

$$f(xe^{2\pi i}) = (xe^{2\pi i})^a = x^a e^{2a\pi i} = f(x)e^{2a\pi i}$$

An interesting question would be to what extent a function is defined through its jump behaviour: In this case: Assume that a function $f(z)$ is analytic for all $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $f(xe^{2\pi i}) = f(x)e^{2a\pi i}$ for $x \in \mathbb{R}_{\geq 0}$. What can we tell about f? For one thing, if we define a function $g(z) = z^{-a}f(z)$, then $g(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ as well and for $x \in \mathbb{R}_{\geq 0}$

$$g(xe^{2\pi i}) = (xe^{2\pi i})^{-a} f(xe^{2\pi i}) = x^{-a}e^{-2a\pi i}f(x)e^{2a\pi i} = x^{-a}f(x) = g(x)$$

So $g(z)$ is analytic for $z \in \mathbb{C}$, except for possibly a singularity for $z = 0$. Hence we introduce some local behaviour of g. Assume that for $z \to 0$ $g(z) = \mathcal{O}\left(z^k\right)$, where $k \in \mathbb{Z}$. Then $h(z) = z^{-k}g(z)$ is entire and imposing asymptotic behaviour for $z \to \infty$ pretty much fixes f.

Another example:

Example 1.1.2. Let $f(z)$ be a function that is analytic for $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and for $x \in \mathbb{R}_{\geq 0}$ $f(xe^{2\pi i}) = f(x) + C$, $C \in \mathbb{C}$. Again: What can be said about

f? Define $g(z) = f(z) - \frac{C}{2\pi i}\log z$. Then

$$\begin{aligned} g(xe^{2\pi i}) &= f(xe^{2\pi i}) - \frac{C}{2\pi i}\log\left(xe^{2\pi i}\right) = f(x) + C - \frac{C}{2\pi i}\cdot 2\pi i - \frac{C}{2\pi i}\log x \\ &= f(x) + C - C - \frac{C}{2\pi i}\log x = g(x) \end{aligned}$$

So again $g(x)$ is analytic and imposing local behaviour and large z asymptotics consequently gives you a unique expression for f.

Now that we have a basic feeling about the interplay between functions and their jump behaviour, let's try something a little more ambitious:

Example 1.1.3. Let γ be a curve of finite length in the complex plane (see Figure 1.1) and $f(z)$ some function that is analytic on $\mathbb{C} \setminus \gamma$. Let $f_+(z)$ be $f(z)$ when z approaches γ from its left hand side with respect to its orientation and let $f_-(z)$ be $f(z)$ when z approaches γ from its right hand side (see Figure 1.1). Basically, if you picture yourself walking on the curve γ in the direction of its orientation, then on your left hand you have the curves $'+'$-side and on your right hand its $'-'$-side.
Find a function f that is analytic on $\mathbb{C} \setminus \gamma$ for which

$$f_+(z) = f_-(z) + a(z) \text{ for } z \in \gamma \tag{1.1.1}$$

where $a(z)$ is some (piecewise) analytic function.
It should be noted that (1.1.1) can be solved for more general functions $a(z)$ (see for example [38] or [57]), but throughout this dissertation (piecewise) analyticity will suffice.
The trick to use here is Cauchy's integral formula:
Let γ be a curve of finite length as in Figure 1.1.
Define

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{a(t)}{t-z}dt$$

We want to discuss the jump behaviour of $f(z)$ on γ, so let $z \in \gamma$. Let γ_+ be a curve that is identical to γ away from z and moves to the $'+'$-side of γ close to z as in Figure 1.1. Also, let γ_- be a curve that is identical to γ away from z and moves to the $'-'$-side of γ close to z as in Figure 1.1.
If z lies at the $'+'$ side of γ_3, then using contour deformation, we may write that

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma_-} \frac{a(t)}{t-z}dt$$

Thus, letting z approach γ from the $'+'$-side, we get

$$f_+(z) = \frac{1}{2\pi i}\int\limits_{\gamma_-} \frac{a(t)}{t-z}dt$$

By the same reasoning, if z lies at the $'-'$-side of γ, then

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma_+} \frac{a(t)}{t-z}dt$$

meaning that if we let z approach from the $'-'$-side

$$f_-(z) = \frac{1}{2\pi i}\int\limits_{\gamma_+} \frac{a(t)}{t-z}dt$$

So

$$f_+(z) - f_-(z) = \frac{1}{2\pi i}\int\limits_{\gamma_- \cup \gamma_+^{-1}} \frac{a(t)}{t-z}dt \tag{1.1.2}$$

Note that $\gamma_- \cup \gamma_+^{-1}$ is a closed contour around z.
Therefore, by Cauchy's integral formula, we get

$$\frac{1}{2\pi i}\int\limits_{\gamma_- \cup \gamma_+^{-1}} \frac{a(t)}{t-z}dt = a(z) \tag{1.1.3}$$

Combining (1.1.2) and (1.1.3), we conclude that

$$f_+(z) - f_-(z) = a(z) \tag{1.1.4}$$

Of course, this solution is by no means unique, as again the complete solution would be $f(z) + h(z)$, where h is some entire function. On a final note: Equation (1.1.4) is known as the Sokhotskii-Plemelj formula (see for example [38] or [57]).

Example 1.1.4. Let $f(z)$ be a function that is analytic for $z \in \mathbb{C} \setminus \gamma$, with γ as in Example 1.1.3 and $f_+(z) = f_-(z)b(z)$ for $z \in \gamma$, where $b(z)$ is some analytic function that is nonzero for $z \in \gamma$. Again: we want to find a function that fulfills this condition.
One way of solving this is simply taking the logarithm of both sides of the jump condition, giving

$$\log f_+(z) = \log f_-(z) + \log b(z)$$

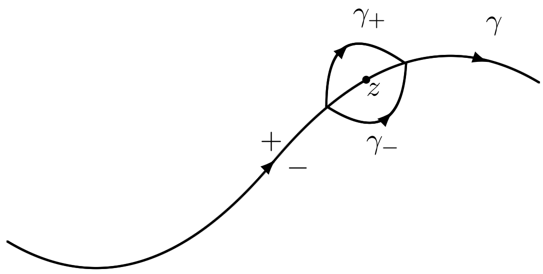

Figure 1.1: The $'+'$-side and $'-'$-side of a curve γ.

at which point we can reuse the Sokhotskii-Plemelj formula from Example 1.1.3 defining

$$\log f(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\log b(t)}{t-z} dt$$

resulting in

$$f(z) = e^{\frac{1}{2\pi i} \int\limits_{\gamma} \frac{\log b(t)}{t-z} dt}$$

So by a Riemann-Hilbert problem, we essentially mean any problem where we are looking for a piecewise analytic function for some imposed jump behaviour and asymptotics. It is worth mentioning that we by no means need to restrict ourselves to scalar functions and can focus on matrix-valued functions instead (which we will start doing from the next section onwards). Our final example in this section will be

Example 1.1.5. [The Szegö function] The Szegö function is (see [64]) a function $D(z)$, that for a specific analytic function $w(z)$ and curve γ on which $w \neq 0$, has the property that $D_+(z)D_-(z) = w(z)$ for $z \in \gamma$. Particularly, we will show how one can use a Riemann-Hilbert problem to find a suitable function for the case that $\gamma = [-1, 1]$: What we want is a function $D(z)$, fulfilling (see for example [43])

- $D(z)$ has no zeros and is analytic for $z \in \mathbb{C} \setminus \gamma$.
- $D_+(z)D_-(z) = w(z)$ for $z \in \gamma$.
- $\lim\limits_{z\to\infty} D(z) = D_\infty \in (0, \infty)$.

Solution: Obviously, we want to get Sokhotskii-Plemelj into this somehow. To that effect, let $E(z) = \log D(z)$, so that

$$E_+(x) = -E_-(x) + \log w(x) \text{ for } x \in [-1, 1]$$

Were it not for the minus-sign in front of E_-, we would be done. So we introduce $F(z) = E(z)(z^2-1)^{-\frac{1}{2}}$, where we choose $(z^2-1)^{\frac{1}{2}}$ to be analytic in $\mathbb{C} \setminus [-1, 1]$. Thus,

$$\begin{aligned}
F_+(x) &= E_+(x)(x^2-1)_+^{-\frac{1}{2}} = (x^2-1)_+^{-\frac{1}{2}}(-E_-(x) + \log w(x)) \\
&= (x^2-1)_+^{-\frac{1}{2}} \log w(x) - (x^2-1)_+^{-\frac{1}{2}} E_-(x) \\
&= -i(1-x^2)^{-\frac{1}{2}} \log w(x) - -E_-(x)(x^2-1)_-^{-\frac{1}{2}} \\
&= -i(1-x^2)^{-\frac{1}{2}} \log w(x) + E_-(x)(x^2-1)_-^{-\frac{1}{2}} \\
&= -i(1-x^2)^{-\frac{1}{2}} \log w(x) + F_-(x)
\end{aligned}$$

So by Sokhotskii-Plemelj, we can choose:

$$F(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\log w(x)}{i\sqrt{1-x^2}(x-z)} dx = \frac{1}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}(z-x)} dx$$

and

$$D(z) = e^{E(z)} = e^{\sqrt{z^2-1}F(z)} = e^{\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}(z-x)} dx}$$

Finally,

$$\lim_{z\to\infty} D(z) = \lim_{z\to\infty} e^{\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}(z-x)} dx} = e^{\frac{1}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx} \in (0, \infty)$$

And with that, all that remains is that D is the unique solution to our Riemann-Hilbert problem:
The related Riemann-Hilbert problem for F would be

- $F(z)$ is analytic on $\mathbb{C} \setminus [-1, 1]$
- $F_+(x) = F_-(x) - i\sqrt{1-x^2} \log w(x)$ for $x \in [-1, 1]$.
- $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{e^{D(z)}}{\sqrt{z^2-1}} = \lim_{z\to\infty} \frac{e^{D_\infty}}{z} = 0$

Assume that the Riemann-Hilbert problem for F has two solutions, F_1 and F_2 say. Then

- $F_1(z) - F_2(z)$ is analytic on $\mathbb{C} \setminus [-1, 1]$,

- $$\begin{aligned}(F_1(x) - F_2(x))_+ &= (F_{1-}(x) - i\sqrt{1-x^2}\log w(x) \\ &\quad - F_{2-}(x))_- + i\sqrt{1-x^2}\log w(x) \\ &= F_{1-}(x) - F_{2-}(x) = (F_1(x) - F_2(x))_- \text{ for } x \in [-1,1]\end{aligned}$$

- $\lim\limits_{z\to\infty} F_1(z) - F_2(z) = 0$

meaning that $F_1(z) - F_2(z)$ is an entire function that is equal to zero at infinity. By Liouville's theorem, that means that $F_1(z) - F_2(z) = 0$, so $F_1(z) = F_2(z)$, so a solution to the Riemann-Hilbert problem for F is unique. Hence the solution to the Riemann-Hilbert problem for D is unique.

1.2 Orthogonal Polynomials

As stated before, our approach for deducing general behaviour of orthogonal polynomials and related objects, is using Riemann-Hilbert techniques. Specifically, we will be using the following Riemann-Hilbert problem, henceforth to be referred to as the Riemann-Hilbert problem for orthogonal polynomials:
Let $\Omega \subset \mathbb{R}$ and w a weight function that is defined upon Ω. Let $Y(z)$ be a 2×2 matrix valued function. The Riemann-Hilbert problem for orthogonal polynomials is then described by

- $Y(z)$ is analytic in $\mathbb{C}\setminus\Omega$.
- $$Y_+(x) = Y_-(x)\begin{pmatrix}1 & w(x)\\ 0 & 1\end{pmatrix} \text{ for } x \in \Omega$$
- $$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix} \text{ as } z\to\infty.$$
- If $\Omega \neq \mathbb{R}$, then some local behaviour around the endpoints is imposed.

The Riemann-Hilbert problem for orthogonal polynomials was formulated by Fokas, Its and Kitaev who showed in their paper [31] that the solution to this problem can be expressed as

$$Y(z) = \begin{pmatrix} \kappa_n^{-1}p_n(z) & \frac{1}{2\pi i\kappa_n}\int\limits_\Omega \frac{p_n(t)w(t)}{t-z}dt \\ -2\pi i\kappa_{n-1}p_{n-1}(z) & -\kappa_{n-1}\int\limits_\Omega \frac{p_{n-1}(t)w(t)}{t-z}dt\end{pmatrix} \tag{1.2.1}$$

where the p_n represents the nth orthonormal polynomial with respect to the weight w and κ_n is its leading coefficient. The beauty of this relation between orthogonal polynomials and the Riemann-Hilbert problem for orthogonal polynomials lies in the fact that what first was an, if elegantly formulated, hard to tackle object when it comes to deduction of asymptotics, is now a matrix-valued problem that can be attacked using a very effective technique called the Deift-Zhou method of steepest descent, which we will further explore in chapter 2.

First we will prove that (1.2.1) is the solution to the Riemann-Hilbert problem for orthogonal polynomials for a specific example that will return repeatedly in this dissertation.

Let $\alpha > -1$, $\beta > -1$, $x_0 \in (-1,1)$ and

$$\nu_{x_0}(z) = \begin{cases} c^2 \text{ for } \operatorname{Re} z \geq x_0 \\ 1 \text{ for } \operatorname{Re} z < x_0 \end{cases} \tag{1.2.2}$$

Furthermore, let

$$w^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta \text{ where } \alpha > -1,\ \beta > -1,\ x \in (-1,1) \tag{1.2.3}$$

be the *Jacobi weight* (see for example [43]) and define

$$w(x) = h(x) w^{\alpha,\beta}(x) \nu_{x_0}(x) \text{ for } x \in (-1,1)$$

where h is a positive real analytic function. The Riemann-Hilbert problem for orthogonal polynomials then becomes:

- $Y(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$
- $$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \text{ for } x \in (-1,x_0) \cup (x_0,1) \tag{1.2.4}$$
- $$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \text{ as } z \to \infty. \tag{1.2.5}$$
- $Y(z)$ has the following behaviour near $z = 1$:

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix}, \text{ if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, \text{ if } \alpha = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \alpha > 0, \end{cases} \tag{1.2.6}$$
$$\text{as } z \to 1,\ z \in \mathbb{C} \setminus [-1,1].$$

- $Y(z)$ has the following behaviour near $z = -1$:

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1|^\beta \\ 1 & |z+1|^\beta \end{pmatrix}, \text{ if } \beta < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, \text{ if } \beta = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \beta > 0, \end{cases} \tag{1.2.7}$$
$$\text{as } z \to -1, \ z \in \mathbb{C} \setminus [-1,1].$$

- $Y(z)$ has the following behaviour near $z = x_0$:

$$Y(z) = \begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix} \tag{1.2.8}$$
$$\text{as } z \to x_0$$

First of all, of course, we will verify (1.2.1):
Judging from the second property of Y, we deduce that

$$\begin{pmatrix} Y_{11+}(x) & Y_{12+}(x) \\ Y_{21+}(x) & Y_{22+}(x) \end{pmatrix} = \begin{pmatrix} Y_{11-}(x) & Y_{12-}(x) \\ Y_{21-}(x) & Y_{22-}(x) \end{pmatrix} \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$$

which means that

$$\begin{pmatrix} Y_{11+}(x) \\ Y_{21+}(x) \end{pmatrix} = \begin{pmatrix} Y_{11-}(x) \\ Y_{21-}(x) \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{12+}(x) \\ Y_{22+}(x) \end{pmatrix} = \begin{pmatrix} Y_{11-}(x) \\ Y_{21-}(x) \end{pmatrix} w(x) + \begin{pmatrix} Y_{12-}(x) \\ Y_{22-}(x) \end{pmatrix}$$

Let's focus on the first column of Y for a spell. As $Y_{11+}(x) = Y_{11-}(x)$ and $Y_{21+}(x) = Y_{21-}(x)$, both Y_{11} and Y_{21} must be entire functions. Since their asymptotics for $z \to \infty$ are $z^n + \mathcal{O}\left(z^{n-1}\right)$ and $\mathcal{O}\left(z^{n-1}\right)$, it turns out that Y_{11} and Y_{21} are in fact polynomials of degrees n and $n-1$. Furthermore, Y_{11} is monic.
Moving on to Y's second column, we see, as Y_{11} and Y_{21} are entire, that

$$\begin{pmatrix} Y_{12+}(x) \\ Y_{22+}(x) \end{pmatrix} = \begin{pmatrix} Y_{11}(x) \\ Y_{21}(x) \end{pmatrix} w(x) + \begin{pmatrix} Y_{12-}(x) \\ Y_{22-}(x) \end{pmatrix}$$

which means that by Sokhotskii Plemelj (see Example 1.1.3)

$$\begin{pmatrix} Y_{12}(z) \\ Y_{22}(z) \end{pmatrix} = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{t-z} \begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} w(t) dt$$

solves the jump condition for the second column of Y.
So what are Y_{11} and Y_{21}? To answer that question, one needs to take the prescribed asymptotics of Y into account. As

$$\frac{1}{t-z} = -\sum_{k=0}^{m-1} \frac{t^k}{z^{k+1}} - \frac{1}{z^{m+1}} \frac{t^{m+1}}{t-z}$$

we see that

$$Y_{12}(z) = -\frac{1}{2\pi i} \sum_{k=0}^{n-1} \frac{1}{z^{k+1}} \int_{-1}^{1} t^k Y_{11}(t) w(t) dt + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty$$

and since we need that, by (1.2.5),

$$Y_{12}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$

for $z \to \infty$, we can conclude that

$$\int_{-1}^{1} t^k Y_{11}(t) w(t) dt = 0$$

for $k < n$.
By a similar argument,

$$\int_{-1}^{1} t^k Y_{21}(t) w(t) dt = 0$$

for $k < n-1$, which means that Y_{11} and Y_{21} are equal to the nth and $n-1$st orthonormal polynomials respectively up to multiplication by a constant. Furthermore, since, again by (1.2.5),

$$Y_{11}(z) = z^n + \mathcal{O}\left(z^{n-1}\right)$$

as $z \to \infty$ and the leading coefficient of the asymptotic expansion of Y_{22} is equal to 1, it must be so that $Y_{11}(z) = \kappa_n^{-1} p_n(z)$ and $Y_{21}(z) = -2\pi i \kappa_{n-1} p_{n-1}(z)$, κ_n being the leading coefficient of p_n.
What still remains is to verify that the imposed local behaviour of the Riemann-Hilbert problem of the second column is fulfilled. This is evident using Lemma 7.2.2 from [1] and the reasoning in the proof of Theorem 2.4 in [43].
So

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{1}{2\pi i \kappa_n} \int_{-1}^{1} \frac{p_n(t) w(t)}{t-z} dt \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{-1}^{1} \frac{p_{n-1}(t) w(t)}{t-z} dt \end{pmatrix} \tag{1.2.9}$$

We should ask ourselves however, whether this is the Riemann-Hilbert problem's only solution or not. To answer this question, we should first check for a solution Y's invertibility:
Observe that $\det(Y(z))$ has no jump, as

$$\begin{aligned}\det(Y_+(x)) &= \det(Y_-(x)) \det \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \\ &= \det(Y_-(x) \cdot 1 \text{ for } x \in (-1, x_0) \cup (x_0, 1)\end{aligned}$$

Furthermore, note that

- For $z \to -1$

$$\det(Y(z)) = \begin{cases} \mathcal{O}(|z+1|^\beta) \text{ for } \beta < 0 \\ \mathcal{O}(\log|z+1|) \text{ for } \beta = 0 \\ \mathcal{O}(1) \text{ for } \beta > 0 \end{cases}$$

  (see (1.2.6))

- For $z \to 1$

$$\det(Y(z)) = \begin{cases} \mathcal{O}(|z-1|^\alpha) \text{ for } \alpha < 0 \\ \mathcal{O}(\log|z-1|) \text{ for } \alpha = 0 \\ \mathcal{O}(1) \text{ for } \alpha > 0 \end{cases}$$

  (see (1.2.7))

- For $z \to x_0$

$$\det(Y(z)) = \mathcal{O}(\log|z - x_0|)$$

  (see (1.2.8))

As $\alpha > -1$ and $\beta > -1$ and the behaviour of $\det(Y(z))$ around $z = x_0$ is logarithmic, the singularities of $\det(Y(z))$ at -1, x_0 and 1 are removable. Lastly, $\det(Y(z)) \to 1$ as $z \to \infty$, so by Liouville $\det(Y(z)) = 1$ for all $z \in \mathbb{C}$ and thus Y is invertible. As such, let Y_1, Y_2 be two solutions of the Riemann-Hilbert problem and define $A(z) = Y_1(z)Y_2(z)^{-1}$. Then

- $A(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$
- For $x \in (-1, 1)$

$$\begin{aligned}A_+(x) &= Y_{1+}(x)Y_{2+}(x)^{-1} \\ &= Y_{1-}(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}^{-1} Y_{2-}(x)^{-1} \\ &= Y_{1-}(x)Y_{2-}(x)^{-1} = A_-(x)\end{aligned}$$

- For $z \to \infty$,

$$\begin{aligned}
A(z) &= Y_1(z)Y_2(z)^{-1} \\
&= \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}^{-1}\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)^{-1} \\
&= \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)^{-1} = I + \mathcal{O}\left(\frac{1}{z}\right)
\end{aligned}$$

This means that $A(z)$ is an entire function that approaches the identity matrix as z tends to infinity. So by Liouville, $A(z) = I$ and therefore $Y_1(z)Y_2(z)^{-1} = I$, which means that $Y_1(z) = Y_2(z)$, thereby proving uniqueness.
And so we have validated the link between p_n and the Riemann-Hilbert problem for orthogonal polynomials.
If our orthonormal polynomials are related to the Riemann-Hilbert problem for orthogonal polynomials through (1.2.9), then it is likely that mathematical objects related to orthogonal polynomials can be expressed in terms of the solution Y described in (1.2.9) as well:
Define the kernel $\mathcal{K}_n(x,y)$

$$\mathcal{K}_n(x,y) = \sum_{i=0}^{n-1} p_i(x)p_i(y)$$

Using the Christoffel-Darboux formula

$$\mathcal{K}_n(x,y) = \frac{\kappa_{n-1}}{\kappa_n}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}$$

(see [64]) we write out

$$\frac{1}{2\pi i(x-y)}(0,1)Y_+^{-1}(y)Y_+(x)\begin{pmatrix}1\\0\end{pmatrix}$$

Note that, since $\det(Y) = 1$,

$$\begin{aligned}
(0,1)Y_+^{-1}(y) &= (0,1)\begin{pmatrix} -\kappa_{n-1}\int\limits_{-1}^{1}\frac{p_{n-1}(t)w(t)}{t-y}dt & -\frac{1}{2\pi i\kappa_n}\int\limits_{-1}^{1}\frac{p_n(t)w(t)}{t-y}dt \\ 2\pi i\kappa_{n-1}p_{n-1}(y) & \kappa_n^{-1}p_n(y)\end{pmatrix} \\
&= (2\pi i\kappa_{n-1}p_{n-1}(y), \kappa_n^{-1}p_n(y))
\end{aligned} \tag{1.2.10}$$

and

$$\begin{aligned}
Y_+(x)\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix} \kappa_n^{-1}p_n(x) & \frac{1}{2\pi i\kappa_n}\int\limits_{-1}^{1}\frac{p_n(t)w(t)}{t-x}dt \\ -2\pi i\kappa_{n-1}p_{n-1}(x) & -\kappa_{n-1}\int\limits_{-1}^{1}\frac{p_{n-1}(t)w(t)}{t-x}dt\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\
&= \begin{pmatrix}\kappa_n^{-1}p_n(x) \\ -2\pi i\kappa_{n-1}p_{n-1}(x)\end{pmatrix}
\end{aligned} \tag{1.2.11}$$

Combining (1.2.10) and (1.2.11) gives

$$\begin{aligned}
&\frac{1}{2\pi i(x-y)}(0,1)Y_+^{-1}(y)Y_+(x)\begin{pmatrix}1\\0\end{pmatrix}\\
&=\frac{1}{2\pi i(x-y)}\left(2\pi i\kappa_{n-1}p_{n-1}(y),\kappa_n^{-1}p_n(y)\right)\cdot\begin{pmatrix}\kappa_n^{-1}p_n(x)\\-2\pi i\kappa_{n-1}p_{n-1}(x)\end{pmatrix}\\
&=\frac{\kappa_{n-1}}{\kappa_n}\frac{p_n(x)p_{n-1}(y)-p_n(y)p_{n-1}(x)}{x-y}\\
&=\mathcal{K}_n(x,y)
\end{aligned}$$

Secondly, to find a usable expression for the recurrence coefficients, we will borrow an elegant trick from [27]:
Define the solution to the Riemann-Hilbert problem for orthogonal polynomials to be $Y^{(n)}$.
Then

- $Y^{(n+1)}(z)Y^{(n)}(z)^{-1}$ is entire, as the jumps of $Y^{(n)}(z)$ and $Y^{(n+1)}(z)$ cancel out.
- $Y^{(n+1)}(z)Y^{(n)}(z)^{-1}=\left(I+\mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\left(I+\mathcal{O}\left(\frac{1}{z}\right)\right)^{-1}$ as $z\to\infty$.

So

$$Y^{(n+1)}(z)Y^{(n)}(z)^{-1}=\begin{pmatrix}z+\mathcal{O}(1)&\mathcal{O}(1)\\\mathcal{O}(1)&0\end{pmatrix}\quad\text{as } z\to\infty$$

Due to $Y^{(n+1)}(z)Y^{(n)}(z)^{-1}$ being an analytic function on $\mathbb{C}$, we have that

$$Y^{(n+1)}(z)Y^{(n)}(z)^{-1}=\begin{pmatrix}z-b_n&\frac{a_n}{2\pi i\kappa_{n-1}^2}\\c_n&0\end{pmatrix}\tag{1.2.12}$$

for certain constants a_n, b_n and c_n. Note that for this choice of constants the (1,1)-entry of (1.2.12) gives exactly (7.0.1). Furthermore, because

$$Y^{(n+1)}(z)=\begin{pmatrix}z-b_n&\frac{a_n}{2\pi i\kappa_{n-1}^2}\\c_n&0\end{pmatrix}Y^{(n)}(z)\tag{1.2.13}$$

and taking the (2,1)-entry of $Y^{(n+1)}(z)$ into account , we find that

$$-2\pi i\kappa_n p_n(z)=c_n\kappa_n^{-1}p_n(z)$$

which means that $c_n=-2\pi i\kappa_n^2$.
Now write for $z\to\infty$

$$Y^{(n)}(z)=\left(I+\frac{1}{z}Y_1^{(n)}+\frac{1}{z^2}Y_2^{(n)}+\mathcal{O}\left(\frac{1}{z^3}\right)\right)\begin{pmatrix}z^n&0\\0&z^{-n}\end{pmatrix}\tag{1.2.14}$$

and

$$Y^{(n+1)}(z) = \left(I + \frac{1}{z}Y_1^{(n+1)} + \frac{1}{z^2}Y_2^{(n+1)} + \mathcal{O}\left(\frac{1}{z^3}\right)\right)\begin{pmatrix} z^{n+1} & 0 \\ 0 & z^{-n-1} \end{pmatrix} \tag{1.2.15}$$

Inserting (1.2.14) and (1.2.15) into (1.2.13) gives

$$\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z - b_n & \frac{a_n}{2\pi i \kappa_{n-1}^2} \\ c_n & 0 \end{pmatrix}\left(I + \frac{Y_1^{(n)}}{z} + \frac{Y_2^{(n)}}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \tag{1.2.16}$$

Collecting terms for the (1,2)-entry of (1.2.16), we get

$$\begin{aligned}\mathcal{O}\left(\frac{1}{z^2}\right) &= \left(Y_1^{(n)}\right)_{12} + \frac{a_n}{2\pi i \kappa_{n-1}^2} \\ &\quad + \frac{1}{z}\left(\left(Y_2^{(n)}\right)_{12} - b_n\left(Y_1^{(n)}\right)_{12} + \frac{a_n}{2\pi i \kappa_{n-1}^2}\left(Y_1^{(n)}\right)_{22}\right) + \mathcal{O}\left(\frac{1}{z^2}\right)\end{aligned}$$

giving

$$\left(Y_1^{(n)}\right)_{12} = -\frac{a_n}{2\pi i \kappa_{n-1}^2} \tag{1.2.17}$$

and using (1.2.17),

$$b_n = \frac{\left(Y_2^{(n)}\right)_{12}}{\left(Y_1^{(n)}\right)_{12}} - \left(Y_1^{(n)}\right)_{22}$$

Observe that the right hand side of (1.2.16) is in fact (1.2.1) multiplied from the right with the matrix

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}$$

As such, one can deduce from the (2,1)-entry of (1.2.16) that the leading coefficient of $-2\pi i \kappa_{n-1} p_{n-1}(z)$ must coincide with $\left(Y_1^{(n)}\right)_{21}$, which implies that $\left(Y_1^{(n)}\right)_{21} = -2\pi i \kappa_{n-1}^2$. Substituting this expression into (1.2.17) then shows that

$$a_n = \left(Y_1^{(n)}\right)_{12}\left(Y_1^{(n)}\right)_{21}$$

So to summarise:

Proposition 1.2.1. Let

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{1}{2\pi i \kappa_n} \int\limits_{\mathbb{R}} \frac{p_n(t)w(t)}{t-z} dt \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int\limits_{\mathbb{R}} \frac{p_{n-1}(t)w(t)}{t-z} dt \end{pmatrix}$$

and

$$Y^{(n)}(z) = \left(I + \frac{1}{z} Y_1^{(n)} + \frac{1}{z^2} Y_2^{(n)} + \mathcal{O}\left(\frac{1}{z^3}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

Then

- $$\mathcal{K}_n(x, y) = \frac{1}{2\pi i (x-y)} (0, 1) Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.18}$$

- $$a_n = \left(Y_1^{(n)}\right)_{12} \left(Y_1^{(n)}\right)_{21} \tag{1.2.19}$$

- $$b_n = \frac{\left(Y_2^{(n)}\right)_{12}}{\left(Y_1^{(n)}\right)_{12}} - \left(Y_1^{(n)}\right)_{22} \tag{1.2.20}$$

Equation (1.2.18) is an identity used in most papers on the topic of Riemann-Hilbert methods related to orthogonal polynomials (see for example [15]) and (1.2.19) and (1.2.20) were taken from [19] and [27].
Thus, by working with (1.2.4) we find out out all we need about p_n, a_n, b_n and $\mathcal{K}_n$.
With our Riemann-Hilbert problem in place, the obvious next question would be how we are going to deduce anything about the behaviour of Y? Before satisfying our curiosity in that respect, we will review a couple of examples of other Riemann-Hilbert problems to familiarise ourselves with the material and preparing ourselves for some problems later on:

1.3 Useful Related Riemann-Hilbert Problems

The main tool in studying the Riemann-Hilbert problem for orthogonal polynomials is the so-called Deift-Zhou steepest descent method for Riemann-Hilbert problems, which allows (1.2.4) to be transformed into a problem with a solution that 'looks like the identity matrix' (an ambiguity we will clarify later on), which means that if we transform back from this solution that 'looks like the identity matrix' to the original problem, we will know the asymptotics of Y, as the transformation used will be explicit. However, for each individual $Y(z)$ specific Riemann-Hilbert problems called 'Parametrices' will appear that require a rather ad hoc solution. To minimise the amount of computation during the analysis later on and obtain some extra exercise in working with Riemann-Hilbert problems, we will formulate them here.

1.3.1 A Parametrix

The following Riemann-Hilbert Problem is of the utmost importance in Deift-Zhou steepest descent analysis (see chapter 2, or [19]), if only for the fact that it is used, in some form, in almost every paper on the topic. Thus, without further ado:

Proposition 1.3.1. Let

$$\begin{cases} M(z) \text{ is analytic in } \mathbb{C} \setminus [a,b] \\ M_+(x) = M_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } x \in (a,b) \\ M(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases}$$

Then, for $\rho(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}}$

$$M(z) = \begin{pmatrix} \frac{\rho(z)+\rho(z)^{-1}}{2} & \frac{\rho(z)-\rho(z)^{-1}}{2i} \\ \frac{\rho(z)-\rho(z)^{-1}}{-2i} & \frac{\rho(z)+\rho(z)^{-1}}{2} \end{pmatrix}$$

is a solution.

Proof. The basic idea you need in order to solve for M, is to come up with a transformation that changes the problem into a problem with a diagonal jump. That way, our 2×2 problem will reduce to a 1×1 variant, decreasing the difficulty of finding a solution significantly.
Thus, define

$$\widehat{M}(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} M(z) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Then for $x \in (a,b)$

$$\begin{aligned} \widehat{M}_+(x) &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} M_+(x) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} M_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{aligned} \tag{1.3.1}$$

Because

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we can rewrite (1.3.1) as

$$\begin{aligned}
\widehat{M}_+(x) &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} M_-(x) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \widehat{M}_-(x) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \widehat{M}_-(x) \cdot \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \widehat{M}_-(x) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\end{aligned}$$

Choosing $\widehat{M}_{12}(z) = \widehat{M}_{21}(z) = 0$ we find that

$$\widehat{M}(z) = \begin{pmatrix} \widehat{M}_{11}(z) & 0 \\ 0 & \widehat{M}_{22}(z) \end{pmatrix}$$

and

$$\begin{cases} \widehat{M}_{11+}(x) = i\widehat{M}_{11-}(x) \\ \widehat{M}_{22+}(x) = -i\widehat{M}_{22-}(x) \end{cases}$$

Observe that if

$$\widehat{M}_{11+}(x) = i\widehat{M}_{11-}(x)$$

then

$$\log \widehat{M}_{11+}(x) = \frac{1}{2}\pi i + \log \widehat{M}_{11-}(x)$$

Thus, by Example 1.1.3,

$$\begin{aligned}
\log \widehat{M}_{11}(z) &= \frac{1}{2\pi i} \int_a^b \frac{\frac{1}{2}\pi i}{t - z} dt \\
&= \frac{1}{4} \int_a^b \frac{1}{t - z} dt \\
&= \frac{1}{4} \left(\log(b - z) - \log(a - z)\right) \\
&= \log \left(\frac{z - b}{z - a}\right)^{\frac{1}{4}}
\end{aligned}$$

and

$$\widehat{M}_{11}(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}} = \rho(z)$$

and by the same argument

$$\widehat{M}_{22}(z) = \left(\frac{z-b}{z-a}\right)^{-\frac{1}{4}} = \rho(z)^{-1}$$

Consequently

$$\widehat{M}(z) = \begin{pmatrix} \rho(z) & 0 \\ 0 & \rho(z)^{-1} \end{pmatrix}$$

and

$$\begin{aligned} M(z) &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \widehat{M}(z) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \rho(z) & 0 \\ 0 & \rho(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\rho(z)+\rho(z)^{-1}}{2} & \frac{\rho(z)-\rho(z)^{-1}}{2i} \\ \frac{\rho(z)-\rho(z)^{-1}}{-2i} & \frac{\rho(z)+\rho(z)^{-1}}{2} \end{pmatrix} \end{aligned}$$

□

1.3.2 The Airy Riemann-Hilbert Problem

The Airy Riemann-Hilbert problem regularly appears in articles that deal with Deift-Zhou steepest descent analysis (see for example [19], [15] and [45]) as a local parametrix (see chapter 2 for a definition) and will be used in chapter 4.
The Airy function Ai is defined through

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int\limits_{C} e^{zt-\frac{t^3}{3}}\, dt$$

where $C = C_1$ (see Figure 1.2) and is a solution to the differential equation

$$y''(z) = zy(z)$$

The main results that we will need about $\mathrm{Ai}(z)$ are Theorem A.2.1 and Theorem A.2.2 (see the appendix).
We express the Airy Riemann-Hilbert problem as follows:
Let

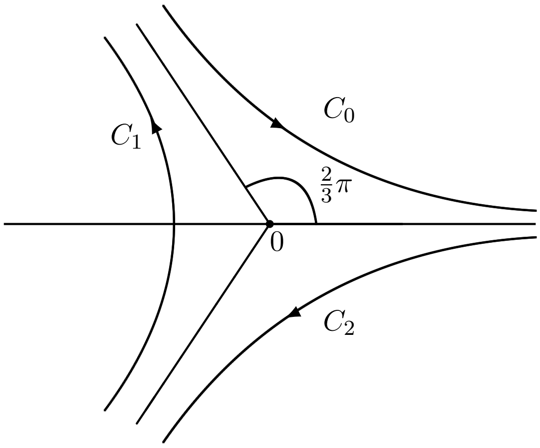

Figure 1.2: Sketch of curves for which the integral representation of $\mathrm{Ai}(z)$ is well defined

- $I = \left\{ z \in \mathbb{C} \mid 0 < \arg z < \frac{2}{3}\pi \right\}$
- $II = \left\{ z \in \mathbb{C} \mid \frac{2}{3}\pi < \arg z < \pi \right\}$
- $III = \left\{ z \in \mathbb{C} \mid \pi < \arg z < \frac{4}{3}\pi \right\}$
- $IV = \left\{ z \in \mathbb{C} \mid \frac{4}{3}\pi < \arg z < 2\pi \right\}$

Then

- $A : \mathbb{C} \setminus \Sigma_{\mathrm{A}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_A is as in Figure 1.3.
- $A_+(z) = A_-(z)J_A(z)$, where $J_A(z)$ represents the jump matrix as in Figure 1.3.
- For $z \to \infty$

$$A(z) = z^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}} \right) \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\frac{2}{3}z^{\frac{3}{2}}\sigma_3} \tag{1.3.2}$$

  where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  is the third Pauli matrix.

- $A(z)$ remains bounded as $z \to 0$ for $z \in \mathbb{C} \setminus \Sigma_{\mathrm{A}}$.

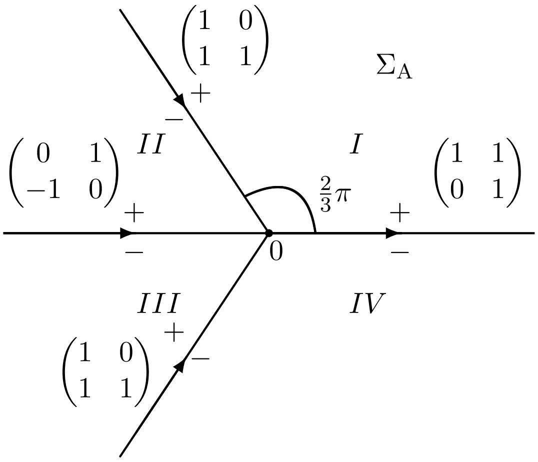

Figure 1.3: Curves on which $A(z)$ has jumps

And has the solution

$$A(z) = \begin{cases} \sqrt{2\pi}\begin{pmatrix} \mathrm{Ai}(z) & -\omega^2\mathrm{Ai}(\omega^2 z) \\ -i\mathrm{Ai}'(z) & i\omega\mathrm{Ai}'(\omega^2 z) \end{pmatrix} & \text{for } z \text{ in sector I} \\ \sqrt{2\pi}\begin{pmatrix} -\omega\mathrm{Ai}(\omega z) & -\omega^2\mathrm{Ai}(\omega^2 z) \\ i\omega^2\mathrm{Ai}'(\omega z) & i\omega\mathrm{Ai}'(\omega^2 z) \end{pmatrix} & \text{for } z \text{ in sector II} \\ \sqrt{2\pi}\begin{pmatrix} -\omega^2\mathrm{Ai}(\omega^2 z) & \omega\mathrm{Ai}(\omega z) \\ i\omega\mathrm{Ai}'(\omega^2 z) & -i\omega^2\mathrm{Ai}'(\omega z) \end{pmatrix} & \text{for } z \text{ in sector III} \\ \sqrt{2\pi}\begin{pmatrix} \mathrm{Ai}(z) & \omega\mathrm{Ai}(\omega z) \\ -i\mathrm{Ai}'(z) & -i\omega^2\mathrm{Ai}'(\omega z) \end{pmatrix} & \text{for } z \text{ in sector IV} \end{cases} \tag{1.3.3}$$

Proof. Obviously, using Theorem A.2.1, A fulfills the jump conditions as posed in Figure 1.3. The Airy function is an analytic function, ensuring boundedness around $z = 0$. What remains is to deduce the asymptotics of $A(z)$ for $z \to \infty$:
From (A.2.2) we learn that for $-\pi < \arg z < \pi$

$$\mathrm{Ai}(z) = \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}}\left(1 + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \tag{1.3.4}$$

We may differentiate (A.2.2) to obtain

$$\mathrm{Ai}'(z) = -\frac{z^{\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}}\left(1 + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \tag{1.3.5}$$

Consequently, inserting ωz into (1.3.4) and (1.3.5) for z, and taking the roots and their cuts along the negative real line into account when removing the ω from $(\omega z)^{\frac{1}{4}}$, we get

$$\sqrt{2\pi}\begin{pmatrix} \omega \mathrm{Ai}(\omega z) \\ -i\omega^2 \mathrm{Ai}'(\omega z)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{\frac{3}{4}} \frac{e^{\frac{2}{3}z^{\frac{3}{2}}}}{z^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ i\omega^{\frac{9}{4}} z^{\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.6}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{\frac{2}{3}z^{\frac{3}{2}}} z^{-\frac{1}{4}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ z^{\frac{1}{4}} e^{\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.7}$$

for $z \in III \cup IV$ and

$$\sqrt{2\pi}\begin{pmatrix} -\omega \mathrm{Ai}(\omega z) \\ i\omega^2 \mathrm{Ai}'(\omega z)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{z^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ iz^{\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.8}$$

for $z \in II$.
Furthermore

$$\sqrt{2\pi}\begin{pmatrix} -\omega^2 \mathrm{Ai}(\omega^2 z) \\ i\omega \mathrm{Ai}'(\omega^2 z)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\omega^{\frac{9}{4}} \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{z^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ -i\omega^{\frac{3}{4}} z^{\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.9}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i\frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ z^{\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.10}$$

for $z \in I \cup II$ and

$$\sqrt{2\pi}\begin{pmatrix} -\omega^2 \mathrm{Ai}(\omega^2 z) \\ i\omega \mathrm{Ai}'(\omega^2 z)\end{pmatrix} = \begin{pmatrix} e^{\frac{2}{3}z^{\frac{3}{2}}} z^{-\frac{1}{4}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right) \\ iz^{\frac{1}{4}} e^{\frac{2}{3}z^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\end{pmatrix} \tag{1.3.11}$$

for $z \in III$.
Inserting (1.3.6), (1.3.8), (1.3.9) and (1.3.11) into (1.3.3) then shows that $A(z)$ indeed fulfills (1.3.2). $\square$

1.3.3 The Bessel Riemann-Hilbert Problem

The Bessel Riemann-Hilbert problem generally appears when studying edge behaviour of orthogonal polynomials around the boundary points of their interval of orthogonality and was first constructed in [43]. We will use this Riemann-Hilbert problem in the Deift-Zhou steepest descent analysis in chapter 2.
We express the Bessel Riemann-Hilbert problem as follows:

Let $\beta > -1$. We consider a matrix valued function

$$B_\beta : \mathbb{C} \setminus \Sigma_{B_\beta} \to \mathbb{C}^{2\times 2}$$

where Σ_{B_β} is represented as in Figure 1.4, that satisfies the following conditions and is essentially a rotation in the complex plane of the identically named function used in [43]:

- B_β is analytic on $\mathbb{C} \setminus \Sigma_{B_\beta}$,
- B_β has the following jump relations:
  - For $z \in \Sigma_{B_\beta}$

$$B_{\beta+}(z) = B_{\beta-}(z) J_{B_\beta}(z)$$

where $J_{B_\beta}(z)$ represents the jump matrix as in Figure 1.4.

- For $z \to 0$, we have that
  - For $\beta < 0$

$$B_\beta(z) = \mathcal{O}\begin{pmatrix} |z|^{-\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \\ |z|^{-\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \end{pmatrix}$$

  - For $\beta = 0$

$$B_\beta(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| \\ \log|z| & \log|z| \end{pmatrix}$$

  - For $\beta > 0$

$$B_\beta(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \\ |z|^{\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \end{pmatrix} & \text{for } z \in III \text{ (see Figure 1.4)} \\ \mathcal{O}\begin{pmatrix} |z|^{-\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \\ |z|^{-\frac{1}{2}\beta} & |z|^{-\frac{1}{2}\beta} \end{pmatrix} & \text{for } z \in I \cup II \text{ (see Figure 1.4)} \end{cases}$$

- For $z \to \infty$

$$B_\beta(z) = (-\pi^2 z)^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(z^{-\frac{1}{2}}\right)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3} \tag{1.3.12}$$

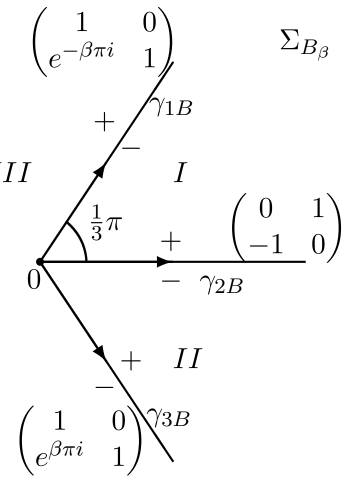

Figure 1.4: Curves on which B has jumps

which has as a solution

$$B_\beta(z) = \begin{cases} \begin{pmatrix} I_\beta\left((-z)^{\frac{1}{2}}\right) & -\frac{i}{\pi}K_\beta\left((-z)^{\frac{1}{2}}\right) \\ -\pi i(-z)^{\frac{1}{2}}I'_\beta\left((-z)^{\frac{1}{2}}\right) & -(-z)^{\frac{1}{2}}K'_\beta\left((-z)^{\frac{1}{2}}\right) \end{pmatrix} & \text{for } z \in III \\ \begin{pmatrix} \frac{1}{2}H^{(1)}_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H^{(2)}_\beta\left(z^{\frac{1}{2}}\right) \\ \frac{1}{2}\pi - iz^{\frac{1}{2}}H^{(1)\prime}_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\beta\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3} & \text{for } z \in II \\ \begin{pmatrix} \frac{1}{2}H^{(2)}_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H^{(1)}_\beta\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(1)\prime}_\beta\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3} & \text{for } z \in I \end{cases} \tag{1.3.13}$$

where I_β, K_β, $H^{(1)}_\beta$ and $H^{(2)}_\beta$ are the modified Bessel functions and the Hankel functions of order β respectively, as in [43] and all power functions have cuts along the negative real axis (where appropriate).

Proof. We will test B_β as a solution to the Bessel Riemann-Hilbert problem. Note that because of (1.3.13)

$$B_\beta(z) = \begin{cases} \begin{pmatrix} \frac{1}{2}H^{(1)}_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H^{(2)}_\beta\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(1)\prime}_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\beta\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3} & \text{for } z \in II \\ \begin{pmatrix} \frac{1}{2}H^{(2)}_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H^{(1)}_\beta\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(1)\prime}_\beta\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\alpha\pi i\sigma_3} & \text{for } z \in I \end{cases}$$

Thus B_β solves the jump along the positive real axis. We will verify the asymptotics for $z \in II$ and leave out the details for the $z \in I$, or $z \in III$,

as the method does not really differ per region.
So, by Theorem A.3.1 we find

$$H_\beta^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{1.3.14}$$

and

$$H_\beta^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{1.3.15}$$

We may differentiate (1.3.14) and (1.3.15) to obtain

$$\left(H_\beta^{(1)}\right)'(z) = i\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right)$$

and analogously

$$\left(H_\beta^{(2)}\right)'(z) = -i\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right)$$

So for $z \in II$

$$\begin{aligned} B_\beta(z) &= \begin{pmatrix} \frac{1}{2}H_\beta^{(1)}\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H_\beta^{(2)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(1)}\right)'\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(2)}\right)'\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3} \\ &= \left(-\pi^2 z\right)^{-\frac{1}{4}\sigma_3} e^{-\frac{1}{4}\pi i\sigma_3}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{z^{\frac{1}{2}}}\right)\right) e^{iz^{\frac{1}{2}}\sigma_3} \end{aligned} \tag{1.3.16}$$

Moving on, we find that $z^{-\frac{1}{4}}e^{-\frac{1}{4}\pi i} = (-z)^{-\frac{1}{4}}$ and $iz^{\frac{1}{2}} = (-z)^{\frac{1}{2}}$, so if we insert that into (1.3.16) we can rewrite (1.3.16) as

$$B_\beta(z) = (-\pi^2 z)^{-\frac{1}{4}\sigma_3}\left(I+\mathcal{O}\left(\frac{1}{z^{\frac{1}{2}}}\right)\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3}$$

What remains is to prove that $B_\beta(z)$ is the solution for $z \in III$.
Note that when we cross γ_{1B}, the solution in III should be

$$\begin{aligned} B_\beta(z) &= \begin{pmatrix} \frac{1}{2}H_\beta^{(2)}\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H_\alpha^{(1)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H_\beta^{(2)\prime}\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}H_\beta^{(1)\prime}\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3}\begin{pmatrix} 1 & 0 \\ e^{-\beta\pi i} & 1\end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(H_\beta^{(1)}\left(z^{\frac{1}{2}}\right)+H_\beta^{(2)}\left(z^{\frac{1}{2}}\right)\right) & \frac{1}{2}H_\beta^{(1)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}\left(\left(H_\beta^{(1)}\right)'\left(z^{\frac{1}{2}}\right)+\left(H_\beta^{(2)}\right)'\left(z^{\frac{1}{2}}\right)\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(1)}\right)'\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3} \end{aligned}$$

Using that $J_\beta(z) = \frac{1}{2}\left(H_\beta^{(1)}(z) + H_\beta^{(2)}(z)\right)$ (see (A.3.3) and (A.3.4)), we can rewrite our matrix as

$$B_\beta(z) = \begin{pmatrix} J_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H_\beta^{(1)}\left(z^{\frac{1}{2}}\right) \\ -\pi i z^{\frac{1}{2}} J'_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(1)}\right)'\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\beta\pi i\sigma_3} \tag{1.3.17}$$

However, when we cross γ_3 instead, following the prescribed jump, we should get

$$\begin{aligned} B_\beta(z) &= \begin{pmatrix} \frac{1}{2}H_\beta^{(1)}\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H_\beta^{(2)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(1)}\right)'\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(2)}\right)'\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\beta\pi i} & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} J_\beta\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H_\beta^{(2)}\left(z^{\frac{1}{2}}\right) \\ -\pi i z^{\frac{1}{2}} J'_\beta\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}\left(H_\beta^{(2)}\right)'\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\beta\pi i\sigma_3} \end{aligned} \tag{1.3.18}$$

So for $B_\beta(z)$ to be analytic for $z \in III$, we need to somehow equate (1.3.17) and (1.3.18).

We will show how this can be done for the first column of $B_{\beta 11}$, as the strategy works essentially the same for the second one.

Observe that through the power series representation of J_β (see (A.3.1)), the power series representation of I_β and analytic continuation of I_β allows us to write

$$B_{11}(z) = J_\beta\left(z^{\frac{1}{2}}\right) e^{-\frac{1}{2}\beta\pi i} = I_\beta\left(e^{-\frac{1}{2}\pi i} z^{\frac{1}{2}}\right) \text{ for } z \in III \text{ and } \operatorname{Im} z > 0$$

As for $z \in III$ with $\operatorname{Im} z > 0$ the equality

$$e^{-\frac{1}{2}\pi i} z^{\frac{1}{2}} = \left(e^{\pi i} z\right)^{\frac{1}{2}}$$

holds, we have that

$$B_{11}(z) = I_\beta\left(\left(e^{\pi i} z\right)^{\frac{1}{2}}\right) \text{ for } z \in III \text{ and } \operatorname{Im} z > 0$$

Analogously,

$$B_{11}(z) = I_\beta\left(\left(e^{-\pi i} z\right)^{\frac{1}{2}}\right) \text{ for } z \in III \text{ and } \operatorname{Im} z < 0$$

As $I_\beta\left(z^{\frac{1}{2}}\right)$ has its cut along the negative real axis, $I_\beta\left((-z)^{\frac{1}{2}}\right)$ is analytic within the desired region, providing us with a valid solution for $B_{11}(z)$ and as a first column of B

$$\begin{pmatrix} I_\beta\left(z^{\frac{1}{2}}\right) \\ -\pi i z^{\frac{1}{2}} I'_\beta\left(z^{\frac{1}{2}}\right) \end{pmatrix}$$

Going through the definitions of $H_\beta^{(1)}$, $H_\beta^{(2)}$ and K_β (see the appendix) it will be no trouble at this point to verify the validity of the second column of B_β. $\square$

1.3.4 The Confluent Hypergeometric Riemann-Hilbert Problem

The Confluent Hypergeometric Riemann-Hilbert Problem was first discovered in [40], streamlined in [32] and generalised in [33]. What will be discussed here is a rotated version of the problem as formulated in [32].
Before discussing the actual problem, we first need some basic identities:
Let $\phi(a,c;z)$ and $\psi(a,c;z)$ be solutions to the confluent hypergeometric equation

$$zy''(z) + (c-z)y'(z) - ay(z) = 0$$

where a, c constant, as defined in [30] through

$$\phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}dt$$

and

$$\psi(a,c;z) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\alpha}} t^{a-1}(1+t)^{c-a-1}e^{-zt}dt$$

The basic results that we will need in this section are that

- $$\phi(a,c;z) = \frac{e^{\mp\pi i(c-a)}\Gamma(c)}{\Gamma(a)} e^z \psi(a,c;e^{\mp\pi i}z) + \frac{e^{\pm\pi i a}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;z) \tag{1.3.19}$$

  (see Lemma A.6.1)

- $$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!(c)_n} \tag{1.3.20}$$

  where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ (see Theorem A.6.5)

- For $z \to \infty$

$$\begin{aligned}\phi(a,c;z) = {} & \frac{e^{\pm a\pi i}z^{-a}\Gamma(c)}{\Gamma(c-a)} \left(\sum_{k=0}^{n} \frac{(1+a-c)_k (a)_k}{k! z^k} + \mathcal{O}\left(\frac{1}{z^{n+1}}\right)\right) \\ & + \frac{z^{a-c}\Gamma(c)}{\Gamma(a)} e^z \left(\sum_{k=0}^{m} \frac{(1-a)_k (c-a)_k}{k! z^k} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right)\right)\end{aligned} \tag{1.3.21}$$

  (see Theorem A.6.5)

- For $z \to 0$

$$\psi(a,c;z) = \begin{cases} \mathcal{O}(z^{1-c}) \text{ for } \operatorname{Re} c > 1 \\ \mathcal{O}(\log z) \text{ for } c = 1 \end{cases} \tag{1.3.22}$$

(see Theorem A.6.5)

- For $z \to \infty$

$$\psi(a,c;z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (1+a+c)_n}{n! z^n} \tag{1.3.23}$$

(see Theorem A.6.5)

The Confluent Hypergeometric Riemann-Hilbert problem can be expressed as follows:
Let $c > 0$ and $c \neq 1$ and $\lambda = \frac{i}{\pi} \log c$. Let

- $I = \left\{ z \in \mathbb{C} \mid 0 < \arg z < \frac{1}{4}\pi \right\}$
- $II = \left\{ z \in \mathbb{C} \mid \frac{1}{4}\pi < \arg z < \frac{3}{4}\pi \right\}$
- $III = \left\{ z \in \mathbb{C} \mid \frac{3}{4}\pi < \arg z < \pi \right\}$
- $IV = \left\{ z \in \mathbb{C} \mid \pi < \arg z < \frac{5}{4}\pi \right\}$
- $V = \left\{ z \in \mathbb{C} \mid \frac{5}{4}\pi < \arg z < \frac{7}{4}\pi \right\}$
- $VI = \left\{ z \in \mathbb{C} \mid \frac{7}{4}\pi < \arg z < 2\pi \right\}$

Let I, II, III, IV, V and VI be separated by the halflines Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and Γ_6 as in Figure 1.5. Let $\Sigma_{C_c} = \bigcup_{i=1}^{6} \Gamma_i$ as in Figure 1.5.
We consider a matrix valued function $C_c : \mathbb{C} \setminus \Sigma_{C_c} \to \mathbb{C}^{2\times 2}$ that satisfies the following conditions:

- $C_c(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_{C_c}$ (see Figure 1.5).
-

$$\begin{aligned}
C_{c+}(z) &= C_{c-}(z) \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \text{ for } z \in \Gamma_4 \\
C_{c+}(z) &= C_{c-}(z) \begin{pmatrix} 0 & c^{-1} \\ -c & 0 \end{pmatrix} \text{ for } z \in \Gamma_3 \\
C_{c+}(z) &= C_{c-}(z) \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \text{ for } z \in \Gamma_2 \cup \Gamma_6 \\
C_{c+}(z) &= C_{c-}(z) \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ for } z \in \Gamma_1 \cup \Gamma_5
\end{aligned}$$

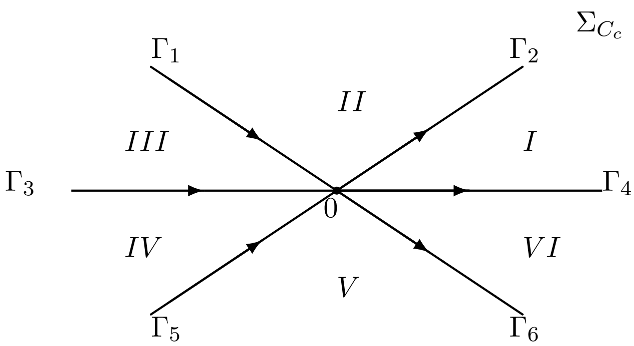

Figure 1.5: Curves on which C_c has jumps

- For $z \to \infty$

$$C_c(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3} c^{\sigma_3} e^{-iz\sigma_3} \text{ for Im } z > 0 \qquad (1.3.24)$$

$$C_c(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{iz\sigma_3} \text{ for Im } z \leq 0 \qquad (1.3.25)$$

where the cut of z^λ is chosen along the negative real axis.

- For z close to 0

$$C_c(z) = \mathcal{O}\left(\log(z)\right)$$

The Confluent Hypergeometric Riemann-Hilbert problem has the following solution:

- For $z \in I$

$$C_c(z) = \begin{pmatrix} c^{-1}\psi(\lambda, 1; 2e^{\frac{1}{2}\pi i}z) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \\ -c^{-1}\frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda, 1; 2e^{\frac{1}{2}\pi i}z) & \psi(-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

- For $z \in II$

$$C_c(z) = \begin{pmatrix} \Gamma(1-\lambda)\phi(\lambda, 1; 2iz) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \\ \Gamma(1+\lambda)\phi(1+\lambda, 1; 2iz) & \psi(-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

- For $z \in III$

$$C_c(z) = \begin{pmatrix} c\psi(\lambda, 1; 2e^{-\frac{3}{2}\pi i}z) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \\ -c\frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda, 1; 2e^{-\frac{3}{2}\pi i}z) & \psi(-\lambda, 1; 2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

- For $z \in IV$

$$C_c(z) = \begin{pmatrix} c\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{\frac{3}{2}\pi i}z) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ -c\psi(-\lambda,1;2e^{\frac{3}{2}\pi i}z) & \frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

- For $z \in V$

$$C_c(z) = \begin{pmatrix} \Gamma(1-\lambda)\phi(\lambda,1;2iz) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ \Gamma(1+\lambda)\phi(1+\lambda,1;2iz) & \frac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

- For $z \in VI$

$$C_c(z) = \begin{pmatrix} c^{-1}\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{-\frac{1}{2}\pi i}z) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ -c^{-1}\psi(-\lambda,1;2e^{-\frac{1}{2}\pi i}z) & \frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3}$$

Proof. Using (1.3.19) to verify the jump behaviour and (1.3.20), (1.3.21), (1.3.22) and (1.3.23) to check the local behaviour around $z = 0$ and the asymptotic behaviour for $z \to \infty$, the proof easily follows. □

And with that, we can move on to the next chapter: The Deift-Zhou steepest descent analysis.

Chapter 2

The Deift-Zhou steepest descent analysis

2.1 Introduction

Deift-Zhou steepest descent analysis is undisputedly a powerful tool (see for example [15], [22], [23], [27], [28], [40], [43], [44] and many, many more) when it comes to deducing asymptotics for Riemann-Hilbert problems and therefore, looking back at the previous chapter, orthogonal polynomials.[1]
This chapter will serve as a quick overview of the basic techniques.
As was said before in chapter 1, we will study an example of steepest descent analysis first as performed in [32], where the orthogonal polynomials studied were polynomials orthogonal with respect to a weight function

$$w(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}\nu_{x_0}(x)$$

on the interval $[-1, 1]$, where $\alpha > -1$, $\beta > -1$, h is a positive and analytic function and

$$\nu_{x_0}(x) = \begin{cases} 1 \text{ if } x < x_0 \\ c^2 \text{ if } x \geq x_0 \end{cases}$$

where $x_0 \in (-1, 1)$.

2.1.1 The Riemann-Hilbert problem

The Riemann-Hilbert problem we will be using as a first encounter with the Deift-Zhou steepest descent analysis is:

- $Y(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$

[1]This chapter does not contain new results.

- $$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \text{ for } x \in (-1, x_0) \cup (x_0, 1)$$

- $$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \text{ as } z \to \infty.$$

- $Y(z)$ has the following behaviour near $z = 1$:

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix}, \text{ if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, \text{ if } \alpha = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \alpha > 0, \end{cases}$$
$$\text{as } z \to 1, \ z \in \mathbb{C} \setminus [-1, 1].$$

- $Y(z)$ has the following behaviour near $z = -1$:

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1|^\beta \\ 1 & |z+1|^\beta \end{pmatrix}, \text{ if } \beta < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, \text{ if } \beta = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \beta > 0, \end{cases}$$
$$\text{as } z \to -1, \ z \in \mathbb{C} \setminus [-1, 1].$$

- $Y(z)$ has the following behaviour near $z = x_0$:

$$Y(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix}$$
$$\text{as } z \to x_0, \ z \in \mathbb{C} \setminus [-1, 1].$$

The unique solution to this problem is (see e.g. [19], or chapter 1)

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{1}{2\pi i \kappa_n} \int\limits_{-1}^{1} \frac{p_n(t) w(t)}{t-z} dt \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int\limits_{-1}^{1} \frac{p_{n-1}(t) w(t)}{t-z} dt \end{pmatrix} \tag{2.1.1}$$

where $p_n(x)$ is the nth degree orthonormal polynomial with respect to the weight $w(x)$, with leading coefficient κ_n.

We will show that Y can be transformed into a matrix valued function $R : \mathbb{C} \to \mathbb{C}^{2\times 2}$, that is, as is Y, dependent on a parameter n and solves

$$\begin{cases} R(z) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z) J_R(z) \text{ for } z \in \Sigma_R \\ \lim_{z\to\infty} R(z) = I \end{cases} \tag{2.1.2}$$

where Σ_R is a finite union of smooth curves in $\mathbb{C}$ and

$$\lim_{n\to\infty} J_R(z) = I \text{ uniformly for every } z \in \Sigma_R \tag{2.1.3}$$

It can be shown for specific examples (see [6], [22], [23], [43], [32] and many more) that a matrix valued function R on which conditions are imposed as described in (2.1.2) and (2.1.3), lies indeed, for large values of n, close to the identity matrix, but we will not get into that here.
Furthermore, we will relate the *normalised reproducing kernel*

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x) p_k(y) w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}$$

to the following kernels:

- The sine kernel

$$\frac{\sin(\pi(x-y))}{\pi(x-y)}$$

- The Bessel kernel

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J_\alpha'(\sqrt{x})}{2(x-y)}, \ x>0,\ y>0$$

  where J_α is the Bessel function of order α.

- The Confluent Hypergeometric kernel

$$\mathbb{K}_c^{CHF}(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda; 2\pi i x); G(\lambda; 2\pi i y)]$$

  where

$$\lambda = \frac{i\log c}{\pi}$$

  and

$$G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$$

  with $\phi(a,c;z)$ as in (A.6.1) and $[f(x);g(y)] = f(x)g(y) - f(y)g(x)$.

We will prove that the following theorem holds:

Theorem 2.1.1.
- For $x \in (-1, x_0) \cup (x_0, 1)$, u, $v \in \mathbb{R}$, as $n \to \infty$

$$\frac{1}{n\xi(x)} K_n \left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)} \right) = \frac{\sin\left(\pi(u - v)\right)}{\pi(u - v)} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.1.4}$$

where $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$

- For $u > 0$, $v > 0$, as $n \to \infty$

$$\frac{1}{2n^2} K_n \left(1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}\right) = \mathbb{J}_\alpha(u, v) + \mathcal{O}\left(\frac{u^{\frac{\alpha}{2}} v^{\frac{\alpha}{2}}}{n}\right) \tag{2.1.5}$$

- For $u > 0$, $v > 0$, as $n \to \infty$

$$\lim_{n\to\infty} K_n \left(-1 + \frac{u}{2n^2}, -1 + \frac{v}{2n^2}\right) = \mathbb{J}_\beta(u, v) + \mathcal{O}\left(\frac{u^{\frac{\beta}{2}} v^{\frac{\beta}{2}}}{n}\right) \tag{2.1.6}$$

- For u, $v \in \mathbb{R}$, as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) = \mathbb{K}_c^{CHF}(u, v) \tag{2.1.7}$$

Remark 2.1.2. Obviously,

$$\mathbb{K}_c^{CHF}(u, u) = \frac{2\nu_0(u)}{c^2 - 1} \left(G'(1 + \lambda; 2\pi iu) G(\lambda; 2\pi u) - G(1 + \lambda; 2\pi iu) G'(\lambda; 2\pi iu) \right)$$

The statements (2.1.4), (2.1.6) and (2.1.5) were proven in [46] by Kuijlaars and Vanlessen for the case that $c = 1$, continuing on Deift-Zhou steepest descent analysis performed in [43]. In [32] it was proven for the case that $c \neq 1$. Statement (2.1.7) was proven by Foulquié Moreno, Martínez-Finkelshtein and Soussa in [32]. Nevertheless, we will prove the theorem, as we need (2.1.7) later on in chapter 5 where we will work with normalised reproducing kernels K_n and the result in [32] relates to $\mathcal{K}_n$ instead of K_n. Furthermore, the parametrices used in our application of Deift-Zhou steepest descent analysis differ slightly from the analysis used in [43] and [32].

2.2 Outline steepest descent analysis

In almost every example of Deift Zhou steepest descent analysis, one is able to dissect the process into a number of five basic steps:

- Step 1. Normalising at infinity. The first step is to normalise the problem for z going to infinity.

- Step 2. Opening of the lenses. With the proper asymptotic behaviour for $z \to \infty$ in place, what remains is to get the correct limit behaviour of the involved parameter. To that event, curves on which the solution has not yet the desired jump behaviour are split into several curves on which the jumps either approach the identity matrix for large parameter values, or are independent of the parameter.

- Step 3. The parametrix away from the endpoints.
  The parametrix away from the endpoints is a solution to the Riemann-Hilbert problem that one gets when one lets the parameter go to infinity. In this thesis, this problem will generally look like the parametrix M discussed in section 1.3.1.

- Step 4. Parametrices near the endpoints. The parametrices near the endpoints, or local parametrices, are solutions to Riemann-Hilbert problems that model the behaviour close to the endpoints and it usually takes some ad hoc reasoning to get through this. And this is where the Airy Riemann-Hilbert problem (see section 1.3.2), the Bessel Riemann-Hilbert problem (see section 1.3.3) and the Confluent Hypergeometric Riemann-Hilbert problem (see section 1.3.4) come into play.

- Step 5. The final transformation. The final transformation is to multiply the solution from step 3 with the inverse of the parametrix away from the endpoints when you are away from the endpoints and close to the endpoints you multiply with the appropriate inverses of the local parametrices. This way, as the parametrices give (approximate) behaviour of the solution from Step 3 within their respective domains, our final problem consequently behaves like the identity for large parameter values.

It should be stressed, though, that whenever the method is applied, there may be some small alterations to provide for certain oddities specific to the particular problem.

2.3 The Deift-Zhou method of steepest descent

As was said before: The goal of the Deift-Zhou method of steepest descent for Riemann-Hilbert problems is to change the original problem into a problem for which all matrices, jump matrices and solutions alike, are asymptotically close to the identity matrix for large n. In this process we can, in our case, discern five steps:

Step 1 $Y \to T$: The first step will be to normalise our problem at infinity. The resulting problem with solution T will be constructed through

$$T(z) = 2^{n\sigma_3} Y(z) \phi(z)^{-n\sigma_3}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the third Pauli matrix and $\phi(z) = z + (z^2-1)^{\frac{1}{2}}$ for $z \in \mathbb{C} \setminus [-1,1]$. The end result will be that T is normalised at infinity and has an oscillatory jump on $[-1,1]$.

Step 2 $T \to S$: In the second step we strive to smoothen the jump conditions on $(-1,x_0) \cup (x_0,1)$. We will split the jumps on these intervals into three jumps each, resulting in six jump matrices: two on curves from -1 to x_0 and from x_0 to 1 in the upper half plane, two on $(-1,x_0)$ and on $(x_0,1)$ and two on curves from -1 to 0 and from 0 to 1 in the lower half plane, which will have, except for the ones on $(-1,x_0) \cup (x_0,1)$, the desired asymptotics. The jump matrix for $x \in (-1,x_0) \cup (x_0,1)$ will be

$$\begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix}$$

Step 3 As was mentioned in Step 3 of section 2.2, we will construct the parametrix away from the endpoints. In this case, the parametrix will be a function

$$N(z) = D_\infty^{\sigma_3} M(z) D(z)^{-\sigma_3}$$

Here M is as in section 1.3.1, $D(z)$ is the Szegö function with respect to the weight function

$$w(x) = h(x)(1-x)^\alpha (1+x)^\beta \text{ with } x \in (-1,x_0) \cup (x_0,1)$$

(see Example 1.1.5) and $D_\infty = \lim_{z\to\infty} D(z)$.

Step 4 Arriving at the fourth step of the Deift-Zhou steepest descent analysis (see section 2.2) we will construct local approximating solutions $P_{-1}(z)$ $P_{x_0}(z)$ and $P_1(z)$, close to -1, x_0 and 1 respectively. P_{-1} and P_1 will be related to the Bessel Riemann-Hilbert problem (see section 1.3.3) and P_{x_0} will be related to the Confluent Hypergeometric Riemann-Hilbert problem (see section 1.3.4).

Step 5 The fifth step will be constructing $R(z)$ by defining $R(z) = S(z)N(z)^{-1}$ away from the endpoints and $R(z) = S(z)P_{-1}(z)^{-1}$ close to -1, $R(z) = S(z)P_{x_0}(z)^{-1}$ close to x_0 and $R(z) = S(z)P_1(z)^{-1}$ close to 1. $R(z)$ will then at last be of the desired form.

After this overview, we can now start our analysis.

2.3.1 The First Step: $Y \to T$

As mentioned before, the first step comes down to normalisation at infinity. For this purpose, our first transformation will be

$$T(z) = 2^{n\sigma_3} Y(z) \phi(z)^{-n\sigma_3} \text{ where } z \in \mathbb{C} \setminus [-1,1] \tag{2.3.1}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\phi(z) = z + (z^2-1)^{\frac{1}{2}}$ for $z \in \mathbb{C} \setminus [-1,1]$.
Evidently, the local behaviour around -1, x_0 and 1 remains the same and the analyticity condition stays unaltered. The changes lie in the jump condition and the asymptotics:
For $x \in (-1, x_0) \cup (x_0, 1)$,

$$\begin{aligned} T_+(x) &= 2^{n\sigma_3} Y_+(x) \phi_+(x)^{-n\sigma_3} \\ &= 2^{n\sigma_3} Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \phi_+(x)^{-n\sigma_3} \\ &= 2^{n\sigma_3} Y_-(x) \phi_-(x)^{-n\sigma_3} \phi_-(x)^{n\sigma_3} \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \phi_+(x)^{-n\sigma_3} \end{aligned} \tag{2.3.2}$$

Note that

$$T_-(x) = Y_-(x)\phi_-(x)^{-n\sigma_3}$$

Thus (2.3.2) can be written as

$$T_+(x) = T_-(x)\phi_-(x)^{n\sigma_3} \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \phi_+(x)^{-n\sigma_3} \tag{2.3.3}$$

Observe that for $x \in (-1,1)$

$$\phi_+(x)\phi_-(x) = (x + i\sqrt{1-x^2})(x - i\sqrt{1-x^2}) = x^2 + (1-x^2) = 1 \tag{2.3.4}$$

Thus, (2.3.4) allows us to write (2.3.3) as

$$T_+(x) = T_-(x) \begin{pmatrix} \phi_+(x)^{-2n} & w(x) \\ 0 & \phi_-(x)^{-2n} \end{pmatrix}$$

For $z \to \infty$

$$\begin{aligned} T(z) &= 2^{n\sigma_3} Y(z) \phi(z)^{-n\sigma_3} \\ &= 2^{n\sigma_3} \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3} \phi(z)^{-n\sigma_3} \end{aligned} \tag{2.3.5}$$

Using that for $z \to \infty$

$$\phi(z) = (2z)^{-n\sigma_3} + \mathcal{O}\left(\frac{1}{z}\right)$$

(2.3.5) becomes

$$\begin{aligned} T(z) &= 2^{n\sigma_3}\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3}(2z)^{-n\sigma_3} \\ &= \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) 2^{n\sigma_3} z^{n\sigma_3}(2z)^{-n\sigma_3} \\ &= I + \mathcal{O}\left(\frac{1}{z}\right) \end{aligned}$$

Thus, we are provided with

- $T(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$
- $$T_+(x) = T_-(x) \begin{pmatrix} \phi_+(x)^{-2n} & w(x) \\ 0 & \phi_-(x)^{-2n} \end{pmatrix} \text{ for } x \in (-1, x_0) \cup (x_0, 1)$$
- $$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty.$$
- $T(z)$ has the following behaviour near $z = 1$:

$$T(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^{\alpha} \\ 1 & |z-1|^{\alpha} \end{pmatrix}, & \text{if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, & \text{if } \alpha = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \alpha > 0, \end{cases}$$
$$\text{as } z \to 1, \ z \in \mathbb{C} \setminus [-1,1].$$

- $T(z)$ has the following behaviour near $z = -1$:

$$T(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1| \\ 1 & |z+1| \end{pmatrix}, & \text{if } \beta < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, & \text{if } \beta = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \beta > 0, \end{cases}$$
$$\text{as } z \to -1, \ z \in \mathbb{C} \setminus [-1,1].$$

- $T(z)$ has the following behaviour near $z = x_0$:

$$T(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix}$$
$$\text{as } z \to x_0,\ z \in \mathbb{C} \setminus [-1,1].$$

2.3.2 The Second Step: $T \to S$

Recall that the ultimate goal of the Deift-Zhou method of steepest descent is to construct a problem for which both the solution and the jump matrices are asymptotically close to the identity matrix for $n \to \infty$. As the first step has given us normalisation at infinity, the obvious next step is to take care of the oscillatory behaviour of our jump matrix. To this end, we will be using an elegant identity in the form of the following matrix factorisation

$$\begin{pmatrix} \phi_+(x)^{-2n} & w(x) \\ 0 & \phi_-(x)^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w(x)^{-1}\phi_-(x)^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ w(x)^{-1}\phi_+(x)^{-2n} & 1 \end{pmatrix}$$

to split the jumps on $(-1,1)$ into three separate ones, establishing lens shaped figures of jump curves (see Figure 2.1) and defining

$$S(z) = \begin{cases} T(z) \text{ for } z \text{ outside of the lenses} \\ T(z)\begin{pmatrix} 1 & 0 \\ -w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \text{ in the upper halves of the lenses} \\ T(z)\begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \text{ in the lower halves of the lenses} \end{cases} \tag{2.3.6}$$

In Figure 2.1, $\gamma_3 = (-1, x_0)$, $\gamma_4 = (x_0, 1)$ and the curves γ_1, γ_2, γ_5 and γ_6 are chosen in such a way that they are contained in some region $U \subset \mathbb{C}$.
U is a region for which

$$h(z)(1-z)^\alpha(1+z)^\beta \text{ for } z \in U \setminus ((-\infty,-1] \cup [1,\infty))$$

has an analytic extension.
Also, we will redefine $\nu_{x_0}(z)$ through

$$\nu_{x_0}(z) = \begin{cases} 1 \text{ if } \operatorname{Re} z < x_0 \\ c^2 \text{ if } \operatorname{Re} z \geq x_0 \end{cases}$$

Furthermore, we choose U in such a way that

$$\operatorname{Re} h(z) > 0 \text{ for } z \in U \setminus ((-\infty,-1] \cup [1,\infty))$$

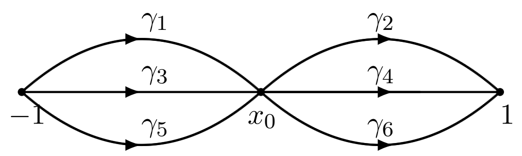

Figure 2.1: Sketch of the curves on which $S(z)$ has jumps.

meaning that for $z \in U \setminus ((-\infty,-1] \cup [1,\infty))$,

$$h(z)(1-z)^{\alpha}(1+z)^{\beta} \text{ for } z \in U \setminus ((-\infty,-1] \cup [1,\infty))$$

is non-zero.
Thus we obtain:

- $S(z)$ is analytic in $\mathbb{C} \setminus \bigcup_{i=1}^{6} \gamma_i$
-
$$S_+(z) = S_-(z) J_S(z) \text{ for } z \in \bigcup_{i=1}^{6} \gamma_i$$

  with

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in \gamma_1 \cup \gamma_2 \\ \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix} \text{ for } z \in \gamma_3 \cup \gamma_4 \\ \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in \gamma_5 \cup \gamma_6 \end{cases}$$

-
$$S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty.$$

- $S(z)$ has the following behaviour near $z = 1$:

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^{\alpha} \\ 1 & |z-1|^{\alpha} \end{pmatrix}, \text{ if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, \text{ if } \alpha = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \alpha > 0, \end{cases}$$
$$\text{as } z \to 1,\ z \in \mathbb{C} \setminus [-1,1].$$

- $S(z)$ has the following behaviour near $z = -1$:

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1|^{\beta} \\ 1 & |z+1|^{\beta} \end{pmatrix}, \text{ if } \beta < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, \text{ if } \beta = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \beta > 0, \end{cases}$$
$$\text{as } z \to -1, \ z \in \mathbb{C} \setminus [-1,1].$$

- $S(z)$ has the following behaviour near $z = x_0$:

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix} & \text{for } z \text{ outside of the lenses} \\ \mathcal{O}\begin{pmatrix} \log|z-x_0| & \log|z-x_0| \\ \log|z-x_0| & \log|z-x_0| \end{pmatrix} & \text{for } z \text{ inside of the lenses} \end{cases}$$
$$\text{as } z \to x_0$$

Note that $\phi(z)$ maps $\mathbb{C} \setminus [-1,1]$ to the outside region of the unit circle, so $J_S(z)$ goes to the identity matrix exponentially fast for $n \to \infty$ for

$$z \in \gamma_1 \cup \gamma_2 \cup \gamma_5 \cup \gamma_6$$

And with that we have all we could wish for, except for the jumps on $(-1,1)$ and the behaviour around the points $z = -1$, $z = x_0$ and $z = 1$. Hence our next step will be to find the so-called parametrix away from the endpoints.

2.3.3 The Third Step: $N(z)$, The Parametrix Away From The Endpoints

$N(z)$, the parametrix away from the endpoints, is a solution to the problem

- $N(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$
-
$$N_+(x) = N_-(x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} \text{ for } x \in (-1,x_0) \cup (x_0,1)$$
-
$$N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty.$$

Note that this Riemann-Hilbert problem looks very much like a problem we have already solved in chapter 1 for the case that $w(z) = 1$ (see Proposition 1.3.1). Let's choose $a = -1$ and $b = 1$ and define N to be

$$N(z) = D_\infty^{\sigma_3} M(z) D(z)^{-\sigma_3}$$

where $D_\infty = \lim_{z\to\infty} D(z)$ and $D(z)$ is the Szegö function (see [43], or Example 1.1.5) with respect to the function $w(z)$ and $M(z)$ is as in section 1.3.1. Thus, $N(z)$ is analytic on $\mathbb{C} \setminus [-1, 1]$.
Furthermore, for $x \in (-1, 1)$

$$\begin{aligned} N_+(x) &= D_\infty^{\sigma_3} M_+(x) D_+(x)^{-\sigma_3} \\ &= D_\infty^{\sigma_3} M_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D_+(x)^{-\sigma_3} \end{aligned} \tag{2.3.7}$$

where we have used the jump property of M.
We can rewrite (2.3.7) as

$$\begin{aligned} N_+(x) &= D_\infty^{\sigma_3} M_-(x) D_-(x)^{-\sigma_3} D_-(x)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D_+(x)^{-\sigma_3} \\ &= N_-(x) \begin{pmatrix} 0 & D_-(x)D_+(x) \\ -(D_-(x)D_+(x))^{-1} & 0 \end{pmatrix} \end{aligned} \tag{2.3.8}$$

Using that

$$D_+(x)D_-(x) = w(x)$$

(see Example 1.1.5), (2.3.8) becomes

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix}$$

and

$$\lim_{z\to\infty} N(z) = \lim_{z\to\infty} D_\infty^{\sigma_3} M(z) D(z)^{-\sigma_3} = D_\infty^{\sigma_3} I D_\infty^{-\sigma_3} = I$$

so both asymptotics and jump behaviour are taken care of.
Thus, we have succesfully found a parametrix away from the endpoints.

2.3.4 The Fourth Step: The Local Parametrices P_{-1}, P_{x_0} and P_1.

Our objective in this step is to find approximations P_{-1}, P_{x_0} and P_1 around $z = -1$, $z = x_0$ and $z = 1$ respectively, called local parametrices, approximating the local behaviour of S. As before, we will be basically following [43] and [32].

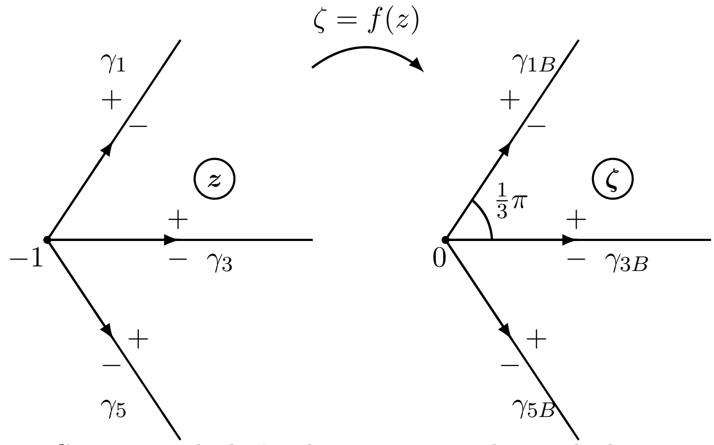

Figure 2.2: Curves on which P_{-1} has jumps together with the curves on which B_β has jumps.

Constructing P_{-1}.

Let's try and construct P_{-1}. After transforming to S, around -1, the jump curves of S look like the left hand side of Figure 2.2. As the caption of Figure 2.2 suggests, we want to construct P_{-1} with the help of B_β, as described in section 1.3.3. Before getting into that, we will first formulate a Riemann-Hilbert problem for P_{-1} within a small surrounding region U_{-1} of -1 that is enclosed by U (see section 2.3.2):

- $P_{-1}(z)$ is analytic in $U_{-1} \setminus (\gamma_1 \cup \gamma_3 \cup \gamma_5)$
- $$P_{-1+}(z) = P_{-1-}(z) J_{P_{-1}}(z) \text{ for } z \in U_{-1} \cap (\gamma_1 \cup \gamma_2 \cup \gamma_3)$$
  with
  $$J_{P_{-1}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in U_{-1} \cap \gamma_1 \\ \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix} \text{ for } z \in U_{-1} \cap \gamma_3 \\ \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in U_{-1} \cap \gamma_5 \end{cases}$$
- $$P_{-1}(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U_{-1}. \tag{2.3.9}$$

- $P_{-1}(z)$ has the following behaviour near $z = -1$:

$$P_{-1}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1|^{\beta} \\ 1 & |z+1|^{\beta} \end{pmatrix}, \text{ if } \beta < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, \text{ if } \beta = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \beta > 0, \end{cases}$$
$$\text{as } z \to -1,\ z \in \mathbb{C} \setminus [-1,1].$$

Here the asymptotic condition for $n \to \infty$ is replacing the one for $z \to \infty$ earlier, as we are looking for local behaviour around -1 and we want our solution to approach $N(z)$ for $n \to \infty$.
To that effect, we express P_{-1} as follows:

$$P_{-1}(z) = E_n(z) B_\beta\left(n^2 f(z)\right) W(z)^{-\sigma_3} (-\phi(z))^{-n\sigma_3} \tag{2.3.10}$$

In (2.3.10) $E_n(z)$ is a yet to be constructed function that is analytic around $z = -1$, $\zeta = f(z)$ defines a conformal mapping that maps the jump curves of P_{-1} to the jump curves of B_β as described in Figure 2.2, which will be specified later on and

$$W(z) = \left(h(z)(1-z)^{\alpha}(-1-z)^{\beta}\right)^{\frac{1}{2}}$$

with $W(x) > 0$ for $x \in U_{-1}$, $x < -1$. Thus,

$$W^2(z) = \begin{cases} e^{-\beta\pi i} w(z) \text{ for } \operatorname{Im} z > 0 \\ e^{\beta\pi i} w(z) \text{ for } \operatorname{Im} z < 0 \end{cases} \tag{2.3.11}$$

and

$$W_+(x) W_-(x) = w(x) \text{ for } x \in (-1, x_0) \tag{2.3.12}$$

Our aim is now to choose f and E_n in such a way that (2.3.10) is indeed a solution to the Riemann-Hilbert problem for P_{-1}.
Observe that, using (2.3.11) and (2.3.12), the jump behaviour of $P_{-1}(z) W(z)^{\sigma_3} \phi(z)^{n\sigma_3}$ is identical to the jump behaviour of $E_n(z) B_\beta\left(n^2 f(z)\right)$, which has the same jump behaviour as $B_\beta\left(n^2 f(z)\right)$, because $E_n(z)$ is entire.
The jump conditions for P_{-1} fulfilled, we will now verify the limit behaviour of P_{-1} for $n \to \infty$:
Recall from (1.3.12) that

$$B_\beta(z) = (-\pi^2 z)^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(z^{-\frac{1}{2}}\right)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3}$$

for $z \to \infty$.
So for $n \to \infty$, (2.3.10) gives

$$P_{-1}(z) = E_n(z)\left(-\pi^2 n^2 f(z)\right)^{-\frac{1}{4}\sigma_3}\left(I + \mathcal{O}\left(\left(n^2 f(z)\right)^{-\frac{1}{2}}\right)\right)$$
$$\cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-n^2 f(z))^{\frac{1}{2}}\sigma_3} W(z)^{-\sigma_3}(-\phi(z))^{-n\sigma_3} \tag{2.3.13}$$

Choose

$$f(z) = -\left(\log\left(-\phi(z)\right)\right)^2 \tag{2.3.14}$$

For $z \to -1$

$$f(z) = (z+1) + \mathcal{O}\left((z+1)^2\right) \tag{2.3.15}$$

Using (2.3.14) and (2.3.15), we can rewrite (2.3.13) as

$$P_{-1}(z) = E_n(z)\left(-\pi^2 n^2 f(z)\right)^{-\frac{1}{4}\sigma_3}\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} W(z)^{-\sigma_3} \tag{2.3.16}$$

Define

$$E_n(z) = N(z)W(z)^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(-\pi^2 n^2 f(z)\right)^{\frac{1}{4}\sigma_3} \tag{2.3.17}$$

Inserting (2.3.17) into (2.3.16) then proves that (2.3.9) holds. However, E_n was assumed to be analytic around $z = -1$, so we still have to check that E_n doesn't have a jump. For this we refer to Proposition 6.5 of [43]. The proof for our case works exactly the same. Thus we have found a valid expression for

$$P_{-1}(z) = E_n(z)B_\beta(n^2 f(z))W(z)^{-\sigma_3}(-\phi(z))^{-n\sigma_3} \tag{2.3.18}$$

where

$$E_n(z) = N(z)W(z)^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(-\pi^2 n^2 f(z)\right)^{\frac{1}{4}\sigma_3} \tag{2.3.19}$$

where

$$f(z) = -(\log\left(-\phi(z)\right))^2 \tag{2.3.20}$$

and

$$W(z) = \left(h(z)(1-z)^\alpha(-1-z)^\beta\right)^{\frac{1}{2}} \tag{2.3.21}$$

with $W(x) > 0$ for $x \in U_{-1}$, $x < -1$.

Constructing P_{x_0}

Next, we will deal with P_{x_0}. As was the case with $P_{-1}(z)$ around $z=-1$, $P_{x_0}(z)$ is to approximate the behaviour of $S(z)$ for z close to x_0. To that end, analogously to what was done for P_{-1}, we will define P_{x_0} within a region U_{x_0} surrounding x_0 and enclosed by U, after constructing the following Riemann-Hilbert problem:

- $P_{x_0}(z)$ is analytic in $U_{x_0} \setminus \bigcup_{i=1}^{6} \gamma_i$
- $$P_{x_0+}(z) = P_{0-}(z) J_{P_0}(z) \text{ for } z \in \left(\bigcup_{i=1}^{6} \gamma_i \right) \cap U_{x_0}$$

  with

  $$J_{P_{x_0}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} & \text{for } z \in (\gamma_1 \cup \gamma_2) \cap U_{x_0} \\ \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix} & \text{for } z \in (\gamma_3 \cup \gamma_4) \cap U_{x_0} \\ \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} & \text{for } z \in (\gamma_5 \cup \gamma_6) \cap U_{x_0} \end{cases}$$

- $$P_{x_0}(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U_{x_0}. \tag{2.3.22}$$

- $P_{x_0}(z)$ has the following behaviour near $z = x_0$:

  $$P_{x_0}(z) = \begin{cases} \begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix} & \text{for } z \text{ outside of the lenses} \\ \begin{pmatrix} \log|z-x_0| & \log|z-x_0| \\ \log|z-x_0| & \log|z-x_0| \end{pmatrix} & \text{for } z \text{ inside of the lenses} \end{cases}$$

Repeating our strategy used for P_{-1}, we write

$$P_{x_0}(z) = E_n(z) C_c(nf(z)) W(z)^{-\sigma_3} \phi(z)^{-n\sigma_3} \tag{2.3.23}$$

where $E_n(z)$ is a function that is analytic around $z = x_0$ and will be used to get the proper limit behaviour on ∂U_{x_0}, C_c is defined as in section 1.3.4,

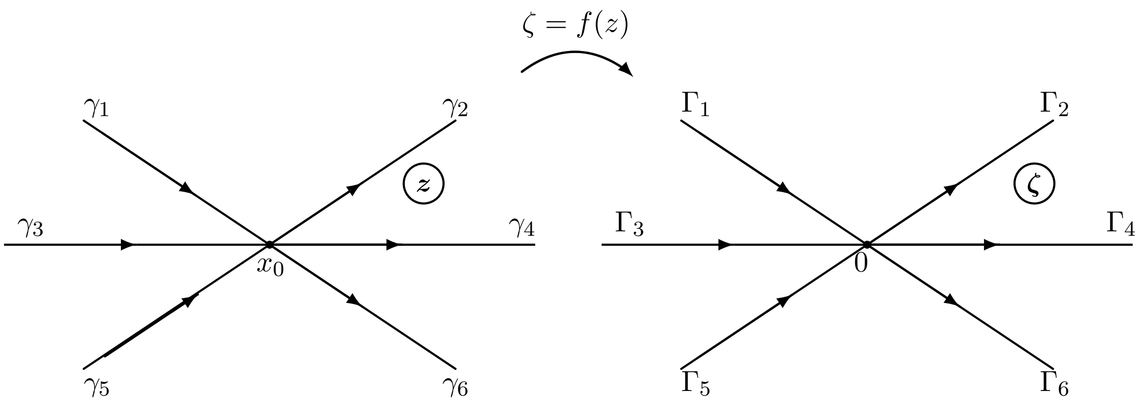

Figure 2.3: Sketch of curves on which C_c has jumps together with the curves on which P_{x_0} has jumps.

$\zeta = f(z)$ is a conformal map that maps the jump curves of P_{x_0} to the jump curves of C_c as in Figure 2.3 that will be specified later on and

$$W(z) = \begin{cases} \sqrt{w(z)c} \text{ for Re } z < x_0 \\ \sqrt{w(z)c^{-1}} \text{ for Re } z \geq x_0 \end{cases}$$

such that $W(x) > 0$ for $x \in (-1, 1)$. Note that $W(z)$ has no discontinuity along Re $z = x_0$.
As was done for P_{-1}, we will now deduce f and E_n by testing the expression (2.3.23) as a solution to the Riemann-Hilbert problem for P_{x_0}. Observe that, because of (2.3.23),

$$P_{x_0}(z)W(z)^{\sigma_3}\phi(z)^{n\sigma_3} = E_n(z)C_c(nf(z))$$

So $P_{x_0}(z)W(z)^{\sigma_3}\phi(z)^{n\sigma_3}$ has the same jump behaviour as $C_c(z)$, as E_n is assumed to be analytic.
Recall from (1.3.24) that

$$C_c(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}c^{\sigma_3}e^{-iz\sigma_3} \text{ for Im } z > 0$$
$$C_c(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{iz\sigma_3} \text{ for Im } z \leq 0$$

Thus, we get that for $z \in \partial U_{x_0}$ the equation (2.3.23) becomes

$$\begin{aligned} P_{x_0}(z) &= E_n(z) C_c(nf(z)) W(z)^{-\sigma_3} \phi(z)^{-n\sigma_3} \\ &= E_n(z) \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) (2nf(z))^{-\lambda\sigma_3} c^{\sigma_3} e^{-inf(z)\sigma_3} \\ &\cdot W(z)^{-\sigma_3} \phi(z)^{-n\sigma_3} \end{aligned} \tag{2.3.24}$$

for Im $z > 0$ and

$$\begin{aligned} P_{x_0}(z) = E_n(z) \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) (2nf(z))^{-\lambda\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{inf(z)\sigma_3} \\ \cdot W(z)^{-\sigma_3} \phi(z)^{-n\sigma_3} \end{aligned} \tag{2.3.25}$$

for Im $z \leq 0$.
Choose

$$f(z) = \begin{cases} \arccos x_0 + i \log \phi(z) \text{ for Im } z > 0 \\ \arccos x_0 - i \log \phi(z) \text{ for Im } z < 0 \end{cases} \tag{2.3.26}$$

where we take the main branch of the logarithm. Note that $f(z)$ is analytic around $z = x_0$: Let $x \in (-1, 1)$. Then

$$\begin{aligned} \phi_+(x) = x + i\sqrt{1-x^2} &= \frac{(x - i\sqrt{1-x^2})(x + i\sqrt{1-x^2})}{x - i\sqrt{1-x^2}} \\ &= \frac{1}{x - i\sqrt{1-x^2}} = \phi_-(x)^{-1} \end{aligned} \tag{2.3.27}$$

Using (2.3.27), we find that for $x \in (-1, 1)$

$$\begin{aligned} f_+(x) &= \arccos x_0 + i \log \phi_+(x) = \arccos x_0 + i \log \phi_-(x)^{-1} \\ &= \arccos x_0 - i \log \phi_-(x) = f_-(x) \end{aligned}$$

proving that $f(z)$ is indeed analytic around $z = x_0$.
Furthermore, for $x \in (-1, 1)$

$$f(x) = f_+(x) = \arccos x_0 + i(\log|\phi_+(x)| + i \arg \phi_+(x)) \tag{2.3.28}$$

From (2.3.27) it follows that $\phi_+(x) = \overline{\phi_-(x)}$ and $\phi_+(x)\phi_-(x) = 1$, so (2.3.28) becomes

$$\begin{aligned} f(x) &= \arccos x_0 + i(\log 1 + i \arg(x + i\sqrt{1-x^2})) \\ &= \arccos x_0 + i(0 + i \arccos x) = \arccos x_0 - \arccos x \end{aligned} \tag{2.3.29}$$

Thus, expanding (2.3.29) as a Taylor series around $x = x_0$, we get

$$f(x) = \frac{1}{\sqrt{1-x_0^2}}(x - x_0) + \mathcal{O}\left((x-x_0)^2\right) \tag{2.3.30}$$

proving that f is indeed a valid conformal map.
So, combining (2.3.26) with (2.3.24) and (2.3.25), we find

$$P_{x_0}(z) = E_n(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right)(2nf(z))^{-\lambda\sigma_3}\, c^{\sigma_3} e^{-in\arccos x_0 \sigma_3} W(z)^{-\sigma_3} \tag{2.3.31}$$

for $\mathrm{Im}\ z > 0$ and

$$P_{x_0}(z) = E_n(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right)(2nf(z))^{-\lambda\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{in\arccos x_0 \sigma_3} W(z)^{-\sigma_3} \tag{2.3.32}$$

for $\mathrm{Im}\ z \leq 0$.
Choose

$$E_n(z) = N(z)\cdot\begin{cases} W(z)^{\sigma_3} e^{in\arccos x_0\sigma_3} c^{-\sigma_3}\,(2nf(z))^{\lambda\sigma_3} & \text{for Im } z > 0 \\ W(z)^{\sigma_3} e^{-in\arccos x_0\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(2nf(z))^{\lambda\sigma_3} & \text{for Im } z \leq 0 \end{cases} \tag{2.3.33}$$

For verification of the analyticity of $E_n(z)$ around $z = x_0$, we refer to the proof of Proposition 15 of [32], which can be applied to our case as well.
Thus we have found a valid expression

$$P_{x_0}(z) = E_n(z) C_c(nf(z)) W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \tag{2.3.34}$$

where

$$f(z) = \begin{cases} \arccos x_0 + i\log\phi(z) & \text{for Im } z > 0 \\ \arccos x_0 - i\log\phi(z) & \text{for Im } z < 0 \end{cases} \tag{2.3.35}$$

and

$$E_n(z) = N(z)\cdot\begin{cases} W(z)^{\sigma_3} e^{in\arccos x_0\sigma_3} c^{-\sigma_3}\,(2nf(z))^{\lambda\sigma_3} & \text{for Im } z > 0 \\ W(z)^{\sigma_3} e^{-in\arccos x_0\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(2nf(z))^{\lambda\sigma_3} & \text{for Im } z \leq 0 \end{cases} \tag{2.3.36}$$

Constructing P_1

As was the case with $P_{-1}(z)$ around $z = -1$ and P_{x_0} around $z = x_0$, P_1 is to approximate $S(z)$ around $z = 1$ in a region $U_1 \subset \mathbb{C}$. Thus we construct a Riemann-Hilbert problem:

- $P_1(z)$ is analytic in $U_1 \setminus (\gamma_2 \cup \gamma_4 \cup \gamma_6)$

- $$P_{1+}(z) = P_{1-}(z) J_{P_1}(z) \text{ for } z \in (\gamma_2 \cup \gamma_4 \cup \gamma_6) \cap U_1$$

  with

$$J_{P_1}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in \gamma_2 \cap U_1 \\ \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix} \text{ for } z \in \gamma_4 \cap U_1 \\ \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\phi(z)^{-2n} & 1 \end{pmatrix} \text{ for } z \in \gamma_6 \cap U_1 \end{cases}$$

- $$P_1(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ and } z \in \partial U_1. \qquad (2.3.37)$$

- $P_1(z)$ has the following behaviour near $z = 1$:

$$P_1(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^{\alpha} \\ 1 & |z-1|^{\alpha} \end{pmatrix}, \text{ if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, \text{ if } \alpha = 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ if } \alpha > 0, \end{cases}$$
$$\text{as } z \to 1,\ z \in \mathbb{C} \setminus [-1, 1].$$

We write

$$P_1(z) = E_n(z)\sigma_3 B_\alpha(n^2 f(z))\sigma_3 W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \qquad (2.3.38)$$

where $E_n(z)$ is an analytic function around $z = 1$, B_α as in section 1.3.3, this time with parameter α, $\zeta = f(z)$ a conformal map that maps the jump curves of P_1 to the jump curves of B_α as in Figure 2.4 and will be specified later on and

$$W(z) = \left((z-1)^{\alpha}(z+1)^{\beta}h(z)\right)^{\frac{1}{2}} \qquad (2.3.39)$$

where $W(x)$ is positive for $x > 1$, $x \in U_1$.
Thus

$$W^2(z) = \begin{cases} e^{\alpha\pi i} w(z) \text{ for } \operatorname{Im} z > 0 \\ e^{-\alpha\pi i} w(z) \text{ for } \operatorname{Im} z < 0 \end{cases} \qquad (2.3.40)$$

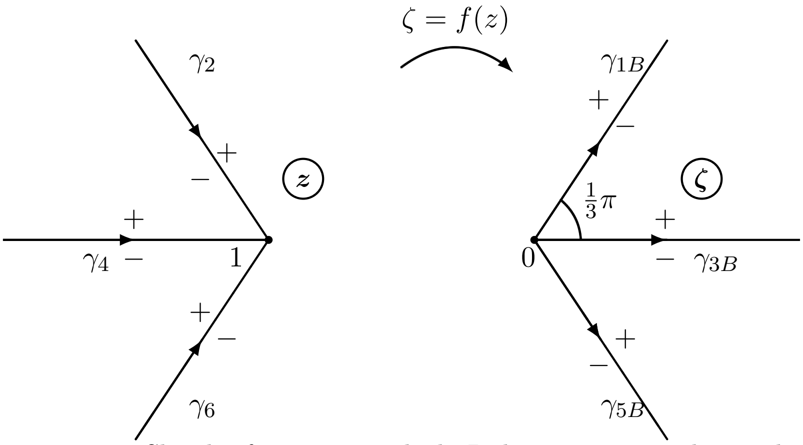

Figure 2.4: Sketch of curves on which P_1 has jumps together with the curves on which B_α has jumps.

and

$$W_+(x)W_-(x) = w(x) \text{ for } x \in (x_0, 1) \tag{2.3.41}$$

Using (2.3.40) and (2.3.41) and taking the jump behaviour of B_α into account, as well as the analyticity of $E_n(z)$, we find that (2.3.38) exactly fulfills the desired jump behaviour for P_1.
From (1.3.12) we learn that

$$B_\alpha(z) = (-\pi^2 z)^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(z^{-\frac{1}{2}}\right)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3}$$

So for $n \to \infty$ and $z \in \partial U_1$

$$\begin{aligned} P_1(z) &= E_n(z)\sigma_3 B_\alpha(n^2 f(z))\sigma_3 W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \\ &= E_n(z)(-\pi^2 n^2 f(z))^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(n^2 f(z))^{\frac{1}{2}}\sigma_3} \\ &\qquad \cdot W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \end{aligned} \tag{2.3.42}$$

Choosing $f(z) = -(\log\phi(z))^2$ allows (2.3.42) to be rewritten as

$$P_1(z) = E_n(z)(-\pi^2 n^2 f(z))^{-\frac{1}{4}\sigma_3} \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} W(z)^{-\sigma_3} \tag{2.3.43}$$

Expanding $f(z)$ as a Taylor series around $z = 1$ gives

$$f(z) = -(z-1) + \mathcal{O}\left((z-1)^2\right)$$

Thus we have verified that f is indeed a suitable conformal map.
If we define E_n through

$$N(z) = E_n(z)(-\pi^2 n^2 f(z))^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} W(z)^{-\sigma_3}$$

Then $P_1(z)$ fulfills the asymptotic condition on ∂U_{-1} for $n \to \infty$ (see (2.3.37)) and E_n is analytic, as can be verified in [43]. Thus we find that

$$P_1(z) = E_n(z)\sigma_3 B_\alpha(n^2 f(z))\sigma_3 W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \tag{2.3.44}$$

where

$$f(z) = -(\log\phi(z))^2$$

where

$$E_n(z) = N(z)W(z)^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (-\pi^2 n^2 f(z))^{\frac{1}{4}\sigma_3}$$

and

$$W(z) = \Big((z-1)^\alpha (z+1)^\beta h(z)\Big)^{\frac{1}{2}}$$

with $W(x) > 0$ for $x > 1$, $x \in U_1$.

2.3.5 The Fifth Step: $S \to R$

As stated before, we define R through

$$R(z) = \begin{cases} S(z)N(z)^{-1} \text{ for } z \in \mathbb{C} \setminus (U_{-1} \cup U_0 \cup U_1) \\ S(z)P_{-1}^{-1}(z) \text{ for } z \in U_{-1} \\ S(z)P_{x_0}^{-1}(z) \text{ for } z \in U_{x_0} \\ S(z)P_1^{-1}(z) \text{ for } z \in U_1 \end{cases} \tag{2.3.45}$$

Thus, R has no jumps within $U_{-1} \cup U_{x_0} \cup U_1$, as the jumps of S cancel out with the jumps of P_{-1}, P_{x_0} and P_1 and no jump on

$$(-1,1) \setminus (U_{-1} \cup U_{x_0} \cup U_1)$$

as the jump of S is nullified by the jump of N. However, three new jumps emerge, namely on ∂U_{-1}, ∂U_{x_0} and ∂U_1 (see Figure 2.5). Furthermore, due to the local behaviour of $P_{-1}(z)$, $P_{x_0}(z)$ and $P_1(z)$ around $z = -1$, $z = x_0$ and $z = 1$ respectively, together with the local behaviour of S around these values, R has no poles at -1, x_0 and 1. The resulting Riemann-Hilbert problem becomes:

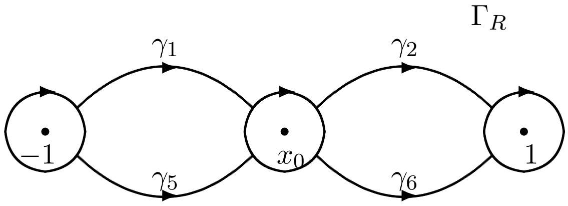

Figure 2.5: Sketch of the curves on which $R(z)$ has jumps.

- $R(z)$ is analytic in $\mathbb{C} \setminus \Gamma_R$ (see Figure 2.5).
- $$R_+(x) = R_-(x) J_R(x)$$
  with
  $$J_R(z) = \begin{cases} N(z)J_S(z)N(z)^{-1} \text{ for } z \in \Gamma_R \setminus (\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1) \\ N(z)P_{-1}^{-1}(z) \text{ for } z \in \partial U_{-1} \\ N(z)P_{x_0}^{-1}(z) \text{ for } z \in \partial U_{x_0} \\ N(z)P_1^{-1}(z) \text{ for } z \in \partial U_1 \end{cases}$$
- $$R(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty.$$

Recall that $J_S(z)$ is exponentially close to the identity matrix uniformly for

$$z \in \Gamma_R \setminus (\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1)$$

and $n \to \infty$, so $N(z)J_S(z)N(z)^{-1}$ is exponentially close to the identity matrix for $z \in \Gamma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1)$.
Analogously to [43] and [32] the jumps on ∂U_{-1}, ∂U_{x_0} and ∂U_1 behave like $I + \mathcal{O}\left(\frac{1}{n}\right)$. Thus, R is indeed the Riemann-Hilbert problem we were looking for. Particularly, we may conclude (see [32]) that

$$R(z) = I + \mathcal{O}\left(\frac{1}{n}\right)$$

2.4 Proof of Theorem 2.1.1

With the Deift-Zhou steepest descent analysis complete, we can now deduce the asymptotic behaviour of

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}} \text{ for } x,\ y \in (-1,1)$$

Recall from Proposition 1.2.1 that

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}(0,1)Y_+^{-1}(y)Y_+(x)\begin{pmatrix}1\\0\end{pmatrix} \tag{2.4.1}$$

where

$$\mathcal{K}_n(x,y) = K_n(x,y)w(x)^{-\frac{1}{2}}w(y)^{-\frac{1}{2}}$$

By (2.3.1)

$$T(z) = 2^{n\sigma_3}Y(z)\phi(z)^{-n\sigma_3}$$

So we can rewrite (2.4.1) as

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}(0,1)\phi_+(y)^{-n\sigma_3}T_+^{-1}(y)T_+(x)\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.4.2}$$

Next, we will use that (see (2.3.6))

$$S(z) = \begin{cases} T(z) \text{ for } z \text{ outside of the lenses} \\ T(z)\begin{pmatrix}1 & 0\\ -w(z)^{-1}\phi(z)^{-2n} & 1\end{pmatrix} \text{ for } z \text{ in the upper halves of the lenses} \\ T(z)\begin{pmatrix}1 & 0\\ w(z)^{-1}\phi(z)^{-2n} & 1\end{pmatrix} \text{ for } z \text{ in the lower halves of the lenses}\end{cases}$$

Thus, we can express (2.4.2) as

$$\begin{aligned}\mathcal{K}_n(x,y) &= \frac{1}{2\pi i(x-y)}(0,1)\phi_+(y)^{-n\sigma_3}\begin{pmatrix}1 & 0\\ -w(y)^{-1}\phi_+(y)^{-2n} & 1\end{pmatrix}S_+^{-1}(y)\\ &\qquad\cdot S_+(x)\begin{pmatrix}1 & 0\\ w(x)^{-1}\phi_+(x)^{-2n} & 1\end{pmatrix}\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}\\ &= \frac{1}{2\pi i(x-y)}(-w(y)^{-1},1)\phi_+(y)^{-n\sigma_3}S_+^{-1}(y) \qquad (2.4.3)\\ &\qquad\cdot S_+(x)\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix}\end{aligned}$$

Now all is in place to start proving our Theorem 2.1.1

2.4.1 Proof of (2.1.4)

We will first set out to prove that for

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}}$$

we have that for $x,\, y \in (-1, x_0) \cup (x_0, 1)$, $u,\, v \in \mathbb{R}$

$$\frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)} + \mathcal{O}\left(\frac{1}{n}\right)$$

Observe that for $z \in (-1,1) \setminus (U_{-1} \cup U_{x_0} \cup U_1)$, due to (2.3.45) we have

$$S(z) = R(z)N(z) \tag{2.4.4}$$

Thus, (2.4.3) can be expressed as

$$\begin{aligned}\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)} &(-w(y)^{-1}, 1)\phi_+(y)^{-n\sigma_3} N_+^{-1}(y) R^{-1}(y) \\ &\cdot R(x) N_+(x)\phi_+(x)^{n\sigma_3} \begin{pmatrix} 1 \\ w(x)^{-1} \end{pmatrix}\end{aligned} \tag{2.4.5}$$

Note that $R(z) = I + \mathcal{O}\left(\frac{1}{n}\right)$, so for some closed curve Γ around z, we see by Cauchy's formula that

$$R'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{R(t) - I}{(t-z)^2} dt = \mathcal{O}\left(\frac{1}{n}\right) \tag{2.4.6}$$

Integrating (2.4.6) from y to x gives

$$R(x) - R(y) = \mathcal{O}\left(\frac{x-y}{n}\right) \tag{2.4.7}$$

Multiplying (2.4.7) from the left by $R(y)^{-1}$ then leads to

$$R^{-1}(y)R(x) - I = \mathcal{O}\left(\frac{x-y}{n}\right) \tag{2.4.8}$$

Using (2.4.8) with (2.4.5) then leads to

$$\begin{aligned}\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)} &(-w(y)^{-1}, 1)\phi_+(y)^{-n\sigma_3} N_+^{-1}(y) \\ &\cdot N_+(x)\phi_+(x)^{n\sigma_3} \begin{pmatrix} 1 \\ w(x)^{-1} \end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned} \tag{2.4.9}$$

Multiplying both sides of (2.4.9) with $w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}$ gives

$$\begin{aligned}K_n(x,y) = \frac{1}{2\pi i(x-y)} &(-1, 1) w(y)^{-\frac{1}{2}\sigma_3} \phi_+(y)^{-n\sigma_3} N_+^{-1}(y) \\ &\cdot N_+(x)\phi_+(x)^{n\sigma_3} w(x)^{\frac{1}{2}\sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned} \tag{2.4.10}$$

Expanding $N_+(x)w(x)^{\frac{1}{2}\sigma_3}$ as a Taylor series around y gives

$$N_+(x)w(x)^{\frac{1}{2}\sigma_3} = N_+(y)w(y)^{\frac{1}{2}\sigma_3} + \mathcal{O}(x-y) \tag{2.4.11}$$

Inserting (2.4.11) into (2.4.10) provides us with

$$K_n(x,y) = \frac{1}{2\pi i(x-y)}(-1,1)\phi_+(y)^{-n\sigma_3}\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} + \mathcal{O}\left(1\right) \tag{2.4.12}$$

From (2.3.26)-(2.3.29) it follows that for $x \in (-1,1)$

$$\log\phi_+(x) = i\arccos x \tag{2.4.13}$$

Inserting (2.4.13) into (2.4.12) gives

$$\begin{aligned}
K_n(x,y) &= \frac{1}{2\pi i(x-y)}(-1,1)e^{-n(\log\phi_+(y))\sigma_3}e^{n(\log\phi_+(x))\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} + \mathcal{O}\left(1\right)\\
&= \frac{1}{2\pi i(x-y)}(-1,1)e^{in(\arccos x-\arccos y)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} + \mathcal{O}\left(1\right)\\
&= \frac{e^{-in(\arccos x-\arccos y)} - e^{in(\arccos x-\arccos y)}}{2\pi i(x-y)} + \mathcal{O}\left(1\right)\\
&= -\frac{\sin\left(n(\arccos x-\arccos y)\right)}{\pi(x-y)} + \mathcal{O}\left(1\right) \qquad (2.4.14)
\end{aligned}$$

Expanding $\arccos x - \arccos y$ as a Taylor series in x gives

$$\arccos x - \arccos y = \frac{1}{\sqrt{1-x^2}}(y-x) + \mathcal{O}\left((y-x)^2\right) \tag{2.4.15}$$

Combining (2.4.15) with (2.4.14) then leads to

$$K_n(x,y) = \frac{\sin\left(n\left(x-y\right)\xi(x)\right)}{\pi(x-y)} + \mathcal{O}\left(1\right) \tag{2.4.16}$$

where

$$\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$$

Replacing x by $x + \frac{u}{n\xi(x)}$ and y by $x + \frac{v}{n\xi(x)}$ in (2.4.16) we conclude that

$$K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) = \frac{n\xi(x)}{\pi}\frac{\sin\left(\pi\left(u-v\right)\right)}{\pi(u-v)} + \mathcal{O}\left(1\right)$$

and thus

$$\frac{1}{n\xi(x)}K_n\left(x + \frac{\pi u}{n\xi(x)}, x + \frac{\pi v}{n\xi(x)}\right) = \frac{\sin\left(\pi\left(u-v\right)\right)}{\pi(u-v)} + \mathcal{O}\left(\frac{1}{n}\right)$$

which proves (2.1.4)

2.4.2 Proof of (2.1.7)

Secondly, we will prove that for

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}} \text{ with } x,\, y \in (-1,1)$$

we have that

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,v) \qquad (2.4.17)$$

for $u,\, v \in \mathbb{R}$, where

$$\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$$

Remember from (2.4.3) that

$$\begin{aligned}\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}&(-w(y)^{-1},1)\phi_+(y)^{-n\sigma_3}S_+^{-1}(y)\\ &\cdot S_+(x)\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix}\end{aligned} \qquad (2.4.18)$$

We will now manipulate S through the parametrix P_{x_0} as described in (2.3.34) as

$$P_{x_0}(z) = E_n(z)C_c(nf(z))W(z)^{-\sigma_3}\phi(z)^{-n\sigma_3} \qquad (2.4.19)$$

where, as we may assume that Im $z > 0$,

$$f(z) = \arccos x_0 + i\log\phi(z)$$

and

$$E_n(z) = N(z)W(z)^{\sigma_3}e^{in\arccos x_0\sigma_3}c^{-\sigma_3}\left(2nf(z)\right)^{\lambda\sigma_3} \text{ for Im } z > 0$$

Note that for $z \in U_{x_0}$, due to (2.3.45)

$$S(z) = R(z)P_{x_0}(z) \qquad (2.4.20)$$

Hence, (2.4.18) can be rewritten as

$$\begin{aligned}\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}&(-w(y)^{-1},1)\phi_+(y)^{-n\sigma_3}P_{x_0+}^{-1}(y)R(y)^{-1}\\ &\cdot R(x)P_{x_0+}(x)\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix}\end{aligned} \qquad (2.4.21)$$

Using (2.4.7) allows (2.4.21) in turn to be rewritten as

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}(-w(y)^{-1},1)\phi_+(y)^{-n\sigma_3}P_{x_0+}^{-1}(y) \cdot P_{x_0+}(x)\phi_+(x)^{n\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.4.22}$$

Using (2.4.19), equation (2.4.22) becomes

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}(-w(y)^{-1},1)W_+(y)^{\sigma_3}C_{c+}(nf(y))^{-1}E_n(y)^{-1} \cdot E_n(x)C_{c+}(nf(x))W_+(x)^{-\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.4.23}$$

Expanding $E_n(y)^{-1}E_n(x)$ as a power series in x around y gives

$$E_n(y)^{-1}E_n(x) = I + \mathcal{O}\left(x-y\right) \tag{2.4.24}$$

Inserting (2.4.24) into (2.4.23) leads to

$$\mathcal{K}_n(x,y) = \frac{1}{2\pi i(x-y)}(-w(y)^{-1},1)W_+(y)^{\sigma_3}C_{c+}(nf(y))^{-1} \cdot C_{c+}(nf(x))W_+(x)^{-\sigma_3}\begin{pmatrix}1\\ w(x)^{-1}\end{pmatrix} + \mathcal{O}\left(1\right) \tag{2.4.25}$$

Multiplying both sides of (2.4.25) with

$$\left(\frac{w(x)}{\nu_{x_0}(x)}\right)^{\frac{1}{2}}\left(\frac{w(y)}{\nu_{x_0(y)}}\right)^{\frac{1}{2}}$$

tells us that

$$K_n(x,y)\nu_{x_0}(x)^{-\frac{1}{2}}\nu_{x_0}(y)^{-\frac{1}{2}} = \frac{1}{2\pi i(x-y)}(-1,1)\left(\frac{\nu_{x_0}(y)}{c}\right)^{-\frac{1}{2}\sigma_3}C_{c+}(nf(y))^{-1} \cdot C_{c+}(nf(x))\left(\frac{\nu_{x_0}(x)}{c}\right)^{\frac{1}{2}\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}\nu_{x_0}(x)^{-\frac{1}{2}}\nu_{x_0}(y)^{-\frac{1}{2}} + \mathcal{O}\left(1\right) \tag{2.4.26}$$

The left hand side of (2.4.26) is an analytic function in x and y, so by analytic continuation we may restrict ourselves to the case that x, $y \in (-1, x_0)$ and consequently that the value of $\nu_{x_0}(x)$ and $\nu_{x_0}(y)$ in the right hand side is equal to 1:

$$K_n(x,y)\nu_{x_0}(x)^{-\frac{1}{2}}\nu_{x_0}(y)^{-\frac{1}{2}} = \frac{1}{2\pi i(x-y)}(-1,1)c^{\frac{1}{2}\sigma_3}C_{c+}(nf(y))^{-1} \cdot C_{c+}(nf(x))c^{-\frac{1}{2}\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} + \mathcal{O}\left(1\right) \tag{2.4.27}$$

Observe that

$$C_{c+}(nf(x))c^{-\frac{1}{2}\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} = c^{-\frac{1}{2}}C_{c+}(nf(x))\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.4.28}$$

Judging from the jump condition on Γ_1 in section 1.3.4, the right hand side of (2.4.28) can be interpreted as the expression one has for C_c in II (see again section 1.3.4). Thus, by the expression for $C_c(z)$ in the region II (see section 1.3.4) we get that (2.4.28) becomes

$$c^{-\frac{1}{2}}\begin{pmatrix}\Gamma(1-\lambda)\phi(\lambda,1;2inf(x))\\ \Gamma(1+\lambda)\phi(1+\lambda,1;2inf(x))\end{pmatrix}e^{-nf(x)} \tag{2.4.29}$$

Analogously,

$$(-1,1)c^{\frac{1}{2}\sigma_3}C_{c+}(nf(y))^{-1} \tag{2.4.30}$$

can be interpreted as

$$c^{-\frac{1}{2}}(0,1)C_{c+}(nf(y))^{-1} \tag{2.4.31}$$

with C_c the expression one would get in II (see again section 1.3.4). Thus, using the expression of $C_c(z)$ for $z \in II$ (see section 1.3.4) we get that (2.4.31) becomes

$$\begin{aligned}&(-1,1)c^{-\frac{1}{2}\sigma_3}C_{c+}(nf(y))^{-1}\\ &= c^{-\frac{1}{2}}\left(-\Gamma(1+\lambda)\phi(1+\lambda,1;2inf(y)),\Gamma(1-\lambda)\phi(\lambda,1,2iz)\right)e^{-nf(y)}\end{aligned} \tag{2.4.32}$$

Thus, inserting (2.4.29) and (2.4.32) into (2.4.26), we get

$$K_n(x,y)\nu_{x_0}(x)^{-\frac{1}{2}}\nu_{x_0}(y)^{-\frac{1}{2}} = \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{2\pi ic(x-y)}[G(1+\lambda,2inf(x));G(\lambda,2inf(y))] + \mathcal{O}\left(1\right) \tag{2.4.33}$$

where $G(a,x) = \phi(a,1;x)e^{-\frac{1}{2}x}$ and $[g(x);h(y)] = g(x)h(y) - g(y)h(x)$.
We know from (2.3.30) that

$$f(x) = \frac{1}{\sqrt{1-x_0^2}}(x-x_0) + \mathcal{O}\left((x-x_0)^2\right) \tag{2.4.34}$$

This means that if we replace x with $x_0 + \frac{u}{n\xi(x_0)}$ and y by $x_0 + \frac{v}{n\xi(x_0)}$, where $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$, then (2.4.33) becomes

$$\begin{aligned}&\frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) \nu_0(u)^{-\frac{1}{2}} \nu_0(v)^{-\frac{1}{2}} \\ &= \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{2\pi i c(u-v)} [G(1+\lambda, 2\pi i u); G(\lambda, 2\pi i v)] + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned} \tag{2.4.35}$$

Thus we can conclude that

$$\begin{aligned}&\frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) \\ &= \frac{\nu_0(u)^{\frac{1}{2}} \nu_0(v)^{\frac{1}{2}} \Gamma(1+\lambda)\Gamma(1-\lambda)}{2\pi i c(x-y)} [G(1+\lambda, 2\pi i u); G(\lambda, 2\pi i v)] + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned} \tag{2.4.36}$$

Rewriting $\Gamma(1+\lambda)\Gamma(1-\lambda)$ completes the proof.

2.4.3 Proof of (2.1.5) and (2.1.6)

After proving (2.1.4) and (2.1.7) we refer to [46] for the proof of (2.1.5) and (2.1.6). While [46] relates to weights that have no jump in x_0, the proof of (2.1.5) and (2.1.6) works the same as in [46].

Chapter 3

Relations between limiting kernels

3.1 Introduction

In the previous chapter we have studied the Riemann-Hilbert problem related to orthogonal polynomials $\{p_k\}_{k=0}^{\infty}$ with respect to a weight function $w(x)$, meaning that

$$\int_{\Omega} p_i(x)p_j(x)w(x)dx = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

where $\Omega \subset \mathbb{R}$ and $w(x) \geq 0$ for $x \in \Omega$.
In the case of chapter 2, we assumed that $\Omega = [-1, 1]$,

$$w(x) = w^{(\alpha,\beta)}(x)\nu_{x_0}(x)h(x) \tag{3.1.1}$$

where $x, x_0 \in (-1, 1)$, $\alpha > -1$, $\beta > -1$, $w^{(\alpha,\beta)}$ is the *Jacobi weight*

$$w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{3.1.2}$$

and

$$\nu_{x_0}(z) = \begin{cases} c^2 \text{ if } \mathrm{Re}\ z \geq x_0 \\ 1 \text{ if } \mathrm{Re}\ z < x_0 \end{cases}$$

Using Deift-Zhou steepest descent analysis, we have learned about the different types of limit behaviour of normalised reproducing kernels

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}} \tag{3.1.3}$$

More specifically, we have related the limit behaviour of K_n to the following three limiting kernels (see 2.1.1)

- The *sine kernel*

$$\frac{\sin(\pi(x-y))}{\pi(x-y)}$$

- The *Bessel kernel*

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}$$

  where J_α is the Bessel function.

- The *Confluent Hypergeometric kernel*

$$\mathbb{K}_c^{CHF}(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda;2\pi ix);G(\lambda;2\pi iy)]$$

  where $\lambda = \frac{i\log c}{\pi}$, $G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$, with $\phi(a,c;z)$ as in (A.6.1) and $[f(x);g(y)] = f(x)g(y) - f(y)g(x)$.

Finding these different types of limit behaviour gives rise to what extent these different kernels are related. In this chapter, we will explore these relations. We will prove that

Theorem 3.1.1. For $s>0$, for all $x,y\in\mathbb{R}$ and $\alpha>-1$,

$$2\pi s\mathbb{J}_\alpha\left(s^2+2\pi xs, s^2+2\pi ys\right) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right) \tag{3.1.4}$$

as $s\to\infty$.

Theorem 3.1.2. For $s\in\mathbb{R}$, for all $x,y\in\mathbb{R}$, $c>0$ and $c\neq 1$,

$$\mathbb{K}_c^{CHF}(x+s,y+s) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s\to\pm\infty$.

Furthermore, we will deduce asymptotic behaviour for a different Bessel kernel

$$\mathbb{J}_\alpha^0(x,y) = \pi\left(\frac{|x|}{x}\right)^\alpha\left(\frac{|y|}{y}\right)^\alpha\sqrt{x}\sqrt{y}\frac{J_{\alpha+\frac{1}{2}}(\pi x)J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x)J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)} \tag{3.1.5}$$

where $\alpha>-\frac{1}{2}$ and $J_{\alpha\pm\frac{1}{2}}$ is the Bessel function of order $\alpha\pm\frac{1}{2}$ (see [3], [34] and Remark 1.2 of [47]). Also, all functions used for $\mathbb{J}_\alpha^0$ have cuts along the negative real line (where applicable). For negative values of x, we will write $x^\alpha = e^{\alpha\pi i}|x|^\alpha$ and $\sqrt{x} = e^{\frac{1}{2}\pi i}\sqrt{|x|}$.
We will prove that:

Theorem 3.1.3. For $s \in \mathbb{R}$, for all $x, y \in \mathbb{R}$,

$$\mathbb{J}_{\alpha}^{0}(s+x, s+y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s \to \pm\infty$.

For background information on the limiting kernels, please see Appendix B.

3.2 Overview of the rest of this chapter

We will prove Theorem 3.1.1 and Theorem 3.1.2 by performing Deift-Zhou steepest descent analysis on the Bessel Riemann-Hilbert problem (see section 1.3.3) and the Confluent hypergeometric Riemann-Hilbert problem (see section 1.3.4) respectively. The proofs of Theorem 3.1.1 and Theorem 3.1.2 will consist of three parts:

- First we will show how the Bessel Riemann-Hilbert problem is related to the Bessel kernel and the Confluent Hypergeometric Riemann-Hilbert problem is related to the Confluent Hypergeometric kernel. This will be done in section 3.3.1 for Theorem 3.1.1 and in section 3.4.1 for Theorem 3.1.2 respectively.

- Secondly we will perform a Deift-Zhou steepest descent analysis by introducing a parameter s and transforming the original problem into a Riemann-Hilbert problem that lies close to the identity matrix for large values of s. For Theorem 3.1.1 this will be explained in section 3.3.2 and for Theorem 3.1.2 this will be explained in section 3.4.2

- Lastly we will use an argument similar to the proof of Theorem 2.1.1 to validate Theorem 3.1.1 in section 3.3.9 and Theorem 3.1.2 in section 3.4.3.

Theorem 3.1.3 will be proven by using identities related to the Bessel function and subsequently using the result of Theorem 3.1.1. This will be achieved in section 3.5.
The techniques used in this chapter are similar to the ones used in [26], [41] and [42].
It should be stressed that while the results in this chapter are new, they can be obtained through easier and more straightforward techniques (see section A.5 for an example). The relevance of chapter 3 however, lies in the method used in the proofs, which could also be applied to other kernels that do not have explicit formulas.

3.3 Proof of Theorem 3.1.1

3.3.1 Relating $\mathbb{J}_\alpha$ to the Bessel Riemann-Hilbert Problem

The formulation of the Bessel Riemann-Hilbert problem used in this chapter differs slightly from the formulation used in section 1.3.3. More specifically, we have rewritten the asymptotic condition for $z \to \infty$ so that the expression $\left(I + \mathcal{O}\left(\frac{1}{z^{\frac{1}{2}}}\right)\right)$ is replaced by $\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)$ and now comes first in the expression of the asymptotic behaviour, which will be of use in the Deift-Zhou steepest descent analysis in section 3.3.2. Note from section 1.3.3 that for any constant matrix C, the function CB_β has exactly the same jump behaviour as B_β. The only thing that changes is the asymptotic condition for $z \to \infty$ which now needs to be multiplied from the left with C. Our first step in this section is to find a suitable matrix C that gives CB_β the desired asymptotic behaviour.
What we would like is a Bessel Riemann-Hilbert problem for which the expression

$$(-\pi^2 z)^{-\frac{1}{4}\sigma_3}\left(I + \mathcal{O}\left(\frac{1}{z^{\frac{1}{2}}}\right)\right) \tag{3.3.1}$$

in (1.3.12) is replaced by

$$\left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}$$

To that end, observe that by Theorem A.3.1

$$H_\beta^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)}\left(1 + \frac{1-4\beta^2}{8iz} - \frac{\left(1-4\beta^2\right)\left(9-4\beta^2\right)}{64z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \tag{3.3.2}$$

and

$$H_\beta^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)}\left(1 - \frac{1-4\beta^2}{8iz} - \frac{\left(1-4\beta^2\right)\left(9-4\beta^2\right)}{64z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \tag{3.3.3}$$

We may differentiate (3.3.2) and (3.3.3) to obtain

$$\left(H_\beta^{(1)}\right)'(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi)}\left(i - \frac{3+4\beta^2}{8z} + i\frac{\left(3+4\beta^2\right)\left(15+4\beta^2\right)}{128z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \tag{3.3.4}$$

and analogously

$$\left(H_\beta^{(2)}\right)'(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\left(z-\frac{1}{2}\beta\pi-\frac{1}{4}\pi\right)} \left(-i - \frac{3+4\beta^2}{8z} - i\frac{\left(3+4\beta^2\right)\left(15+4\beta^2\right)}{128z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \quad (3.3.5)$$

Combining (3.3.2), (3.3.3), (3.3.4) and (3.3.5) with (1.3.13) for

$$0 < \arg z < \frac{\pi}{3}$$

we get that for $z \to \infty$

$$B_\beta(z) = \frac{1}{\sqrt{2}}\pi^{-\frac{1}{2}\sigma_3}(-z)^{-\frac{1}{4}\sigma_3}\left(\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} + \begin{pmatrix} \frac{1-4\beta^2}{8}i & -\frac{1-4\beta^2}{8} \\ -\frac{3+4\beta^2}{8} & \frac{3+4\beta^2}{8}i \end{pmatrix}\frac{1}{z^{\frac{1}{2}}} - \frac{1-4\beta^2}{128\sqrt{2}z}\begin{pmatrix} 9-4\beta^2 & -i\left(9-4\beta^2\right) \\ i\left(15+4\beta^2\right) & -\left(15+4\beta^2\right) \end{pmatrix}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)e^{(-z)^{\frac{1}{2}}\sigma_3}$$

which can be rewritten as

$$B_\beta(z) = \pi^{-\frac{1}{2}\sigma_3}(-z)^{-\frac{1}{4}\sigma_3}\left(I - \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & 1-4\beta^2 \\ 3+4\beta^2 & 0 \end{pmatrix}\frac{1}{z^{\frac{1}{2}}} - \frac{1-4\beta^2}{64\sqrt{2}}\begin{pmatrix} 9-4\beta^2 & 0 \\ 0 & -\left(15+4\beta^2\right) \end{pmatrix}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}e^{(-z)^{\frac{1}{2}}\sigma_3} \quad (3.3.6)$$

Equation (3.3.6) is essentially the asymptotic condition in the Bessel Riemann-Hilbert problem (see section 1.3.3). Observe that (3.3.6) can be rewritten as

$$B_\beta(z) = \pi^{-\frac{1}{2}\sigma_3}\left(I - \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & (-z)^{-\frac{1}{2}}\left(1-4\beta^2\right) \\ (-z)^{\frac{1}{2}}\left(3+4\beta^2\right) & 0 \end{pmatrix}\frac{1}{z^{\frac{1}{2}}} - \frac{1-4\beta^2}{64\sqrt{2}}\begin{pmatrix} 9-4\beta^2 & 0 \\ 0 & -\left(15+4\beta^2\right) \end{pmatrix}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)(-z)^{-\frac{1}{4}\sigma_3} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}e^{(-z)^{\frac{1}{2}}\sigma_3} \quad (3.3.7)$$

Thus, for $0 < \arg z < \frac{\pi}{3}$ and $z \to \infty$

$$B_\beta(z) = \pi^{-\frac{1}{2}\sigma_3}\left(I - \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & (-z)^{-\frac{1}{2}}\left(1-4\beta^2\right) \\ (-z)^{\frac{1}{2}}\left(3+4\beta^2\right) & 0 \end{pmatrix}\frac{1}{z^{\frac{1}{2}}} - \frac{1-4\beta^2}{64\sqrt{2}}\begin{pmatrix} 9-4\beta^2 & 0 \\ 0 & -\left(15+4\beta^2\right) \end{pmatrix}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^{\frac{3}{2}}}\right)\right)(-z)^{-\frac{1}{4}\sigma_3} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}e^{(-z)^{\frac{1}{2}}\sigma_3} \quad (3.3.8)$$

Equation (3.3.8) can then be rewritten as

$$B_\beta(z) = \pi^{-\frac{1}{2}\sigma_3}\left(\begin{pmatrix} 1 & 0 \\ i\left(\frac{3+4\beta^2}{4\sqrt{2}}\right) & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3}$$

Define

$$C = \begin{pmatrix} 1 & 0 \\ -i\left(\frac{3+4\beta^2}{4\sqrt{2}}\right) & 1 \end{pmatrix}\pi^{\frac{1}{2}\sigma_3}$$

Choosing

$$\widehat{B_\beta}(z) = CB_\beta(z)$$

we obtain that for $0 < \arg z < \frac{\pi}{3}$ and $z \to \infty$

$$\widehat{B_\beta}(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3}$$

Repeating this analysis for z outside of the region described by

$$0 < \arg z < \frac{\pi}{3}$$

we find that $\widehat{B_\beta}$ has the desired asymptotic behaviour.
Finally, we change our parameter β to a parameter $\alpha > -1$.
The Bessel Riemann-Hilbert problem expressed in terms of $\widehat{B_\alpha}$ and used in this chapter will be:
Let $\alpha > -1$. We consider a matrix valued function $\widehat{B_\alpha} : \mathbb{C} \setminus \Gamma_{\widehat{B_\alpha}} \to \mathbb{C}^{2\times 2}$ with $\Gamma_{\widehat{B_\alpha}} = \gamma_{1B_\alpha} \cup \gamma_{2B_\alpha} \cup \gamma_{3B_\alpha}$ where $\gamma_{1B_\alpha} = \left\{z \in \mathbb{C} | \arg z = \frac{1}{3}\pi\right\}$, γ_{2B_α} is the positive real axis and $\gamma_{3B_\alpha} = \left\{z \in \mathbb{C} | \arg z = -\frac{1}{3}\pi\right\}$ (see Figure 3.1).
Then $\widehat{B_\alpha}$ satisfies the following conditions:

- $\widehat{B_\alpha}$ is analytic on $\mathbb{C} \setminus \Gamma_{\widehat{B_\alpha}}$, where $\Gamma_{\widehat{B_\alpha}}$ is represented as in Figure 3.1
- $\widehat{B_\alpha}$ has the following jump relations:
  - On γ_{1B_α}

$$\widehat{B_{\alpha+}}(z) = \widehat{B_{\alpha-}}(z)\begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix} \tag{3.3.9}$$

  - On γ_{2B_α}

$$\widehat{B_{\alpha+}}(z) = \widehat{B_{\alpha-}}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3.10}$$

- On γ_{3B_α}:

$$\widehat{B_{\alpha+}}(z) = \widehat{B_{\alpha-}}(z) \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix} \tag{3.3.11}$$

- For $z \to 0$, we have that
  - For $\alpha < 0$

$$\widehat{B_\alpha}(z) = \mathcal{O}\begin{pmatrix} |z|^{-\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \\ |z|^{-\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \end{pmatrix}$$

  - For $\alpha = 0$

$$\widehat{B_\alpha}(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| \\ \log|z| & \log|z| \end{pmatrix}$$

  - For $\alpha > 0$

$$\widehat{B_\alpha}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \\ |z|^{\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } \frac{\pi}{3} < \arg(z) < \frac{5\pi}{3} \\ \mathcal{O}\begin{pmatrix} |z|^{-\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \\ |z|^{-\frac{1}{2}\alpha} & |z|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } 0 < \arg(z) < \frac{\pi}{3} \text{ and } \frac{5\pi}{3} < \arg(z) < 2\pi \end{cases}$$

- For $z \to \infty$

$$\widehat{B_\alpha}(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z)^{\frac{1}{2}}\sigma_3} \tag{3.3.12}$$

which has as a solution

$$\begin{aligned} \widehat{B_\alpha}(z) &= C\begin{pmatrix} I_\alpha\left((-z)^{\frac{1}{2}}\right) & -\frac{i}{\pi}K_\alpha\left((-z)^{\frac{1}{2}}\right) \\ -\pi i(-z)^{\frac{1}{2}}I'_\alpha\left((-z)^{\frac{1}{2}}\right) & -(-z)^{\frac{1}{2}}K'_\alpha\left((-z)^{\frac{1}{2}}\right) \end{pmatrix} \text{ for } \frac{\pi}{3} < \arg(z) < \frac{5\pi}{3} \\ \widehat{B_\alpha}(z) &= C\begin{pmatrix} \frac{1}{2}H^{(1)}_\alpha\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H^{(2)}_\alpha\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(1)\prime}_\alpha\left(z^{\frac{1}{2}}\right) & \frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\alpha\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{\frac{1}{2}\alpha\pi i\sigma_3} \text{ for } \frac{5\pi}{3} < \arg(z) < 2\pi \\ \widehat{B_\alpha}(z) &= C\begin{pmatrix} \frac{1}{2}H^{(2)}_\alpha\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H^{(1)}_\alpha\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(2)\prime}_\alpha\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}}H^{(1)\prime}_\alpha\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\alpha\pi i\sigma_3} \text{ for } 0 < \arg(z) < \frac{\pi}{3} \end{aligned} \tag{3.3.13}$$

where I_α, K_α, $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ are the modified Bessel functions and the Hankel functions respectively, as in [43] and all power functions have cuts along the negative real axis (where appropriate). To link this Riemann-Hilbert problem to the Bessel kernel

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}$$

we will prove the following lemma:

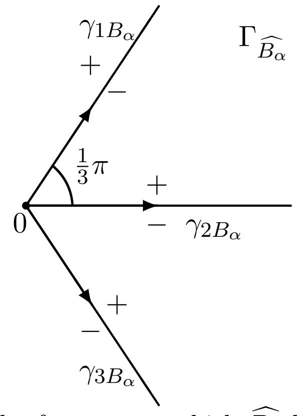

Figure 3.1: Sketch of curves on which $\widehat{B_\alpha}$ has jumps

Lemma 3.3.1. For $x > 0$, $y > 0$

$$\mathbb{J}_\alpha(x,y) = \frac{1}{2\pi i(x-y)} \left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) \widehat{B_{\alpha+}}^{-1}(y)\widehat{B_{\alpha+}}(x) \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix} \tag{3.3.14}$$

Proof. We will start with the expression

$$\frac{1}{2\pi i(x-y)} \left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) \widehat{B_{\alpha+}}^{-1}(y)\widehat{B_{\alpha+}}(x) \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix} \tag{3.3.15}$$

Using basic manipulations with Bessel and Hankel functions we will end up with

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}$$

As we will be needing the inverse of $\widehat{B_\alpha}$, we will first prove that $\det \widehat{B_\alpha} = 1$. Let the jump matrix of $\widehat{B_\alpha}(z)$ be represented by $J_{\widehat{B_\alpha}}(z)$. From the jump conditions on $\widehat{B_\alpha}$ (see (3.3.9)-(3.3.11)) we learn that for $z \in \gamma_{1B_\alpha} \cup \gamma_{2B_\alpha} \cup \gamma_{3B_\alpha}$ the determinant $\det J_{\widehat{B_\alpha(z)}} = 1$, so

$$\begin{aligned} \det \widehat{B_{\alpha+}}(z) &= \det\left(\widehat{B_{\alpha-}}(z) J_{\widehat{B_\alpha}}(z)\right) = \det \widehat{B_{\alpha-}}(z) \det J_{\widehat{B_\alpha}}(z) \\ &= \det \widehat{B_{\alpha-}}(z) \cdot 1 = \det \widehat{B_{\alpha-}}(z) \end{aligned}$$

This means that $\det \widehat{B_\alpha}(z)$ is an entire function.
From (3.3.12) we conclude that for $z \to \infty$

$$\det \widehat{B_\alpha}(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right)$$

Thus, by Liouville, $\det \widehat{B_\alpha}(z) = 1$.
The solution to the Bessel Riemann-Hilbert problem in the sector

$$0 < \arg(z) < \frac{\pi}{3}$$

is

$$\widehat{B_\alpha}(z) = C \begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H_\alpha^{(1)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}} H_\alpha^{(2)\prime}\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}\pi i z^{\frac{1}{2}} H_\alpha^{(1)\prime}\left(z^{\frac{1}{2}}\right) \end{pmatrix} e^{-\frac{1}{2}\alpha\pi i\sigma_3} \tag{3.3.16}$$

(see (3.3.13)).
Thus, the inverse matrix is, as $\det C = 1$,

$$\widehat{B_\alpha}(z)^{-1} = e^{\frac{1}{2}\alpha\pi i\sigma_3} \begin{pmatrix} \frac{1}{2}\pi i z^{\frac{1}{2}} H_\alpha^{(2)\prime}\left(z^{\frac{1}{2}}\right) & -\frac{1}{2}H_\alpha^{(1)}\left(z^{\frac{1}{2}}\right) \\ -\frac{1}{2}\pi i z^{\frac{1}{2}} H_\alpha^{(2)\prime}\left(z^{\frac{1}{2}}\right) & \frac{1}{2}H_\alpha^{(1)}\left(z^{\frac{1}{2}}\right) \end{pmatrix} C^{-1} \tag{3.3.17}$$

Hence, for $y > 0$, using (3.3.17) we find that

$$\begin{aligned} &\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) \widehat{B_{\alpha+}}^{-1}(y) \\ &= \left(-\frac{1}{2}\pi i y^{\frac{1}{2}}\left(H_\alpha^{(2)\prime}\left(y^{\frac{1}{2}}\right) + H_\alpha^{(2)\prime}\left(y^{\frac{1}{2}}\right)\right), \frac{1}{2}\left(H_\alpha^{(1)}\left(y^{\frac{1}{2}}\right) + H_\alpha^{(1)}\left(y^{\frac{1}{2}}\right)\right)\right) C^{-1} \end{aligned} \tag{3.3.18}$$

Similarly, for $x > 0$, using (3.3.16) we find that

$$\widehat{B_{\alpha+}}(x) \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix} = \frac{1}{2}C \begin{pmatrix} \left(H_\alpha^{(2)}\left(x^{\frac{1}{2}}\right) + H_\alpha^{(2)}\left(x^{\frac{1}{2}}\right)\right) \\ -\pi i x^{\frac{1}{2}}\left(H_\alpha^{(2)\prime}\left(x^{\frac{1}{2}}\right) + H_\alpha^{(1)\prime}\left(x^{\frac{1}{2}}\right)\right) \end{pmatrix} \tag{3.3.19}$$

By (A.3.3) and (A.3.4) we have that

$$\begin{aligned} H_\alpha^{(1)}(z) + H_\alpha^{(2)}(z) &= i\frac{e^{-\alpha\pi i}J_\alpha(z) - J_{-\alpha}(z)}{\sin(\alpha\pi)} + i\frac{J_{-\alpha}(z) - e^{\alpha\pi i}J_\alpha(z)}{\sin(\alpha\pi)} \\ &= \frac{i}{\sin(\alpha\pi)}\left(e^{-\alpha\pi i} - e^{\alpha\pi i}\right) J_\alpha(z) = 2J_\alpha(z) \end{aligned} \tag{3.3.20}$$

for $\alpha \notin \mathbb{Z}$. If α is equal to some entire number k, we take the limit where α goes to k instead.
Thus (3.3.18) and (3.3.19) become

$$\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) B_{\alpha+}^{-1}(y) = \left(\pi i y^{\frac{1}{2}} J_\alpha'\left(y^{\frac{1}{2}}\right), J_\alpha\left(y^{\frac{1}{2}}\right)\right) C^{-1} \tag{3.3.21}$$

and

$$B_{\alpha+}(x) \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix} = C \begin{pmatrix} J_\alpha\left(x^{\frac{1}{2}}\right) \\ -\pi i x^{\frac{1}{2}} J_\alpha'\left(x^{\frac{1}{2}}\right) \end{pmatrix} \tag{3.3.22}$$

respectively.
Combining (3.3.21) and (3.3.22) with (3.3.15) then proves the lemma. □

3.3.2 Steepest descent analysis for the Bessel Riemann-Hilbert problem

With Lemma 3.3.1 in place, we want to use Deift-Zhou steepest descent analysis on $\widehat{B_\alpha}$ to deduce proper limit behaviour for

$$\frac{1}{2\pi i(x-y)}\begin{pmatrix}-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\end{pmatrix}\widehat{B_{\alpha+}}^{-1}(y)\widehat{B_{\alpha+}}(x)\begin{pmatrix}e^{\frac{1}{2}\alpha\pi i}\\ e^{\frac{1}{2}\alpha\pi i}\end{pmatrix}$$

3.3.3 The First Step: Transformation $\widehat{B_\alpha} \mapsto A$

The first step will consist of the following shift of variable: Define for $s > 0$ the function

$$A(z) = \widehat{B_\alpha}(z+s) \tag{3.3.23}$$

Let $\Gamma_A = \gamma_{1A} \cup \gamma_{2A} \cup \gamma_{3A}$ where for $i \in \{1,2,3\}$ γ_{iA} is γ_{iB_α} translated a length s to the left. (See Figure 3.2 for Γ_A and Figure 3.1 for $\Gamma_{\widehat{B_\alpha}}$).
Then

- A is analytic on $\mathbb{C} \setminus \Gamma_A$.
- A has the following jump relations:
  - On γ_{1A}
  $$A_+(z) = A_-(z)\begin{pmatrix}1 & 0\\ e^{-\alpha\pi i} & 1\end{pmatrix}$$
  - On γ_{2A}
  $$A_+(z) = A_-(z)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$
  - On γ_{3A}:
  $$A_+(z) = A_-(z)\begin{pmatrix}1 & 0\\ e^{\alpha\pi i} & 1\end{pmatrix}$$
- For $z \to -s$, we have that
  - For $\alpha < 0$
  $$A(z) = \mathcal{O}\begin{pmatrix}|z+s|^{-\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha}\\ |z+s|^{-\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha}\end{pmatrix}$$
  - For $\alpha = 0$
  $$A(z) = \mathcal{O}\begin{pmatrix}\log|z+s| & \log|z+s|\\ \log|z+s| & \log|z+s|\end{pmatrix}$$

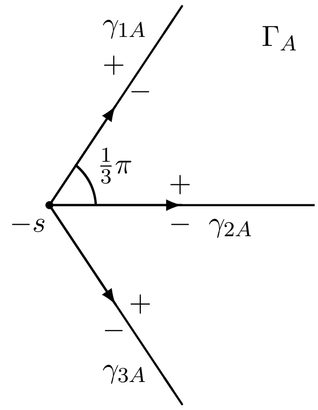

Figure 3.2: Sketch of curves on which A has jumps

– For $\alpha > 0$

$$A(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z+s|^{\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha} \\ |z+s|^{\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } \frac{\pi}{3} < \arg(z+s) < \frac{5\pi}{3} \\ \mathcal{O}\begin{pmatrix} |z+s|^{-\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha} \\ |z+s|^{-\frac{1}{2}\alpha} & |z+s|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } 0 < \arg(z+s) < \frac{\pi}{3} \\ & \text{and } \frac{5\pi}{3} < \arg(z+s) < 2\pi \end{cases}$$

- For $z \to \infty$

$$A(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-z-s)^{\frac{1}{2}}\sigma_3}$$

3.3.4 The Second Step: Transformation $A \mapsto B$

Next, we define, for $s > 0$,

$$B(z) = s^{\frac{1}{4}\sigma_3} A(sz) \tag{3.3.24}$$

Furthermore, let $\Gamma_B = \gamma_{1B} \cup \gamma_{2B} \cup \gamma_{3B}$ where for $i \in \{1, 2, 3\}$ γ_{iB} is γ_{iA} translated horizontally from $-s$ to -1. (See Figure 3.3 for Γ_B and Figure 3.2 for Γ_A).
This is the final step before we can start normalising the Riemann-Hilbert problem for $z \to \infty$, similar to section 2.3.1. Thus, our Riemann-Hilbert problem for B becomes:

- B is analytic on $\mathbb{C} \setminus \Gamma_B$.
- B has the following jump relations:

– On γ_{1B}

$$B_+(z) = B_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}$$

– On γ_{2B}

$$B_+(z) = B_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3.25}$$

– On γ_{3B}:

$$B_+(z) = B_-(z) \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix}$$

- For $z \to -1$, we have that

  – For $\alpha < 0$

$$B(z) = \mathcal{O} \begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix}$$

  – For $\alpha = 0$

$$B(z) = \mathcal{O} \begin{pmatrix} \log|z+1| & \log|z+1| \\ \log|z+1| & \log|z+1| \end{pmatrix}$$

  – For $\alpha > 0$

$$B(z) = \begin{cases} \mathcal{O} \begin{pmatrix} |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } \frac{\pi}{3} < \arg(z+1) < \frac{5\pi}{3} \\ \mathcal{O} \begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } 0 < \arg(z+1)| < \frac{\pi}{3} \\ & \text{and } \frac{5\pi}{3} < \arg(z+1) < 2\pi \end{cases}$$

- For $z \to \infty$

$$B(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{(-s(z+1))^{\frac{1}{2}}\sigma_3}$$

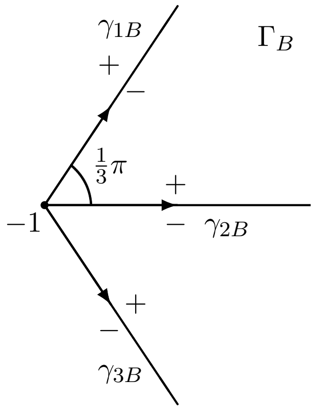

Figure 3.3: Sketch of curves on which B has jumps

3.3.5 The Third Step: Transformation $B \mapsto C$

In this step, we get rid of the exponential behaviour for large z. To that end, let

$$C(z) = B(z)e^{-(-s(z+1))^{\frac{1}{2}}\sigma_3} \tag{3.3.26}$$

Note that for $z \in (-1, \infty)$ (see Figure 3.3)

$$\begin{aligned} C_+(z) &= B_+(z)e^{-(-s(z+1))_+^{\frac{1}{2}}\sigma_3} \\ &= B_+(z)e^{(-s(z+1))_-^{\frac{1}{2}}\sigma_3} \end{aligned}$$

Using (3.3.25), we get

$$\begin{aligned} C_+(z) &= B_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{(-s(z+1))_-^{\frac{1}{2}}\sigma_3} \\ &= C_-(z)e^{(-s(z+1))_-^{\frac{1}{2}}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{(-s(z+1))_-^{\frac{1}{2}}\sigma_3} \end{aligned} \tag{3.3.27}$$

Writing out (3.3.27) then gives

$$C_+(z) = C_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then

- C is analytic on $\mathbb{C} \setminus \Gamma_B$, where Γ_B is represented as in Figure 3.3.
- C has the following jump relations:

– On γ_{1B}

$$C_+(z) = C_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix}$$

– On γ_{2B}

$$C_+(z) = C_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

– On γ_{3B}:

$$C_+(z) = C_-(z) \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix}$$

- For $z \to -1$, we have that

  – For $\alpha < 0$

$$C(z) = \mathcal{O}\begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix}$$

  – For $\alpha = 0$

$$C(z) = \mathcal{O}\begin{pmatrix} \log|z+1| & \log|z+1| \\ \log|z+1| & \log|z+1| \end{pmatrix}$$

  – For $\alpha > 0$

$$C(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } \frac{\pi}{3} < \arg(z+1) < \frac{5\pi}{3} \\ \mathcal{O}\begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } 0 < \arg(z+1) < \frac{\pi}{3} \\ & \text{and } \frac{5\pi}{3} < \arg(z+1) < 2\pi \end{cases}$$

- For $z \to \infty$

$$C(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(-z)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

As the jumps on γ_{1B} and γ_{3B} approach the identity matrix for s going towards infinity, it makes sense to now construct the parametrix away from the endpoint -1.

3.3.6 The Parametrix away from the endpoint

The parametrix away from the endpoint $z = -1$ can be described as a solution N to the problem

- N is analytic on $\mathbb{C} \setminus [-1, \infty)$.
- N has the following jump: For $z \in (-1, \infty)$
$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
- For $z \to \infty$
$$N(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) (-z)^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

A solution to this problem is

$$N(z) = (-z-1)^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

Next, we will take care of the parametrix near the endpoint.

3.3.7 The Parametrix near the endpoint

Within a disk U_δ of radius $\delta > 0$, centered at $z = -1$, we create a parametrix P, fulfilling the following conditions:

- P is analytic on $U_\delta \setminus \Gamma_B$
- P has the following jump conditions:
  - On $\gamma_{1B} \cap U_\delta$
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix}$$
  - On $\gamma_{2B} \cap U_\delta$
  $$P_+(z) = P_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
  - On $\gamma_{3B} \cap U_\delta$:
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix}$$
- For $z \to -1$, we have that

– For $\alpha < 0$

$$P(z) = \mathcal{O}\begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix}$$

– For $\alpha = 0$

$$P(z) = \mathcal{O}\begin{pmatrix} \log|z+1| & \log|z+1| \\ \log|z+1| & \log|z+1| \end{pmatrix}$$

– For $\alpha > 0$

$$P(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } \frac{\pi}{3} < \arg(z+1) < \frac{5\pi}{3} \\ \mathcal{O}\begin{pmatrix} |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \\ |z+1|^{-\frac{1}{2}\alpha} & |z+1|^{-\frac{1}{2}\alpha} \end{pmatrix} & \text{for } 0 < \arg(z+1) < \frac{\pi}{3} \\ & \text{and } \frac{5\pi}{3} < \arg(z+1) < 2\pi \end{cases}$$

- For $s \to \infty$, $z \in \partial U_\delta$

$$P(z) = \left(I + \mathcal{O}\left(\frac{1}{s}\right)\right) N(z) \tag{3.3.28}$$

Conveniently, $P(z) = C(z)$, as all conditions for C within the complex plane apply for P within U_δ. The asymptotic condition for P for $s \to \infty$ holds as well, as it is a direct consequence of the asymptotics of the Bessel Riemann-Hilbert Problem: If $s \to \infty$, then substituting $s(z+1)$ for z in the original asymptotic condition, gives the asymptotics for large s, leading to the final condition for P.
It should be noted that, strictly speaking, P is not a parametrix, as it is not approximating C, it in fact is C.

3.3.8 The Fourth Step: Transformation $C \mapsto R$

Define

$$R(z) = \begin{cases} C(z)P(z)^{-1} \text{ for } z \in U_\delta \\ C(z)N(z)^{-1} \text{ for } z \notin U_\delta \end{cases} \tag{3.3.29}$$

with C as in section 3.3.5.
Let

$$\Gamma_R = \gamma_{1R} \cup U_\delta \cup \gamma_{3R} \cup \partial U_\delta$$

where for $i \in \{1, 2, 3\}$

$$\gamma_{iR} = \gamma_{iB} \cap U_\delta^c$$

with γ_{iB} as in Figure 3.3. See Figure 3.4 for the curves that make up Γ_R.
Then

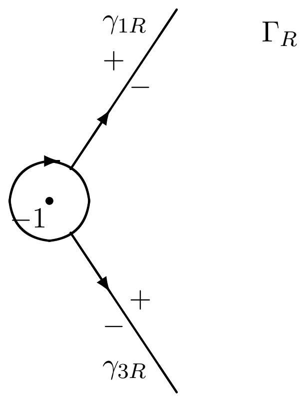

Figure 3.4: Sketch of curves on which R has jumps

- R is analytic on $\mathbb{C} \setminus \Gamma_R$, where Γ_R is represented as in Figure 3.4
- R has the following jump conditions:
  - On γ_{1R}
$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix} N(z)^{-1}$$
  - On ∂U_δ
$$R_+(z) = R_-(z)P(z)N(z)^{-1}$$
  - On γ_{3R}:
$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i - 2(-s(z+1))^{\frac{1}{2}}} & 1 \end{pmatrix} N(z)^{-1}$$
- For $z \to \infty$
$$R(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$$

Note that the jumps of R all go to the identity matrix for $s \to \infty$ at a speed of $\mathcal{O}\left(\frac{1}{s}\right)$: Of the jumps on γ_{1R} and γ_{3R} we already know that they go to the identity matrix exponentially fast, as they are the jumps found on γ_{1B} and γ_{3B} in the Riemann-Hilbert problem for C (see section 3.3.5) conjugated with $N(z)$. The jump matrix on U_δ is $P(z)N(z)^{-1}$ and for $z \in U_\delta$ we have for $s \to \infty$, because of (3.3.28), that

$$P(z)N(z)^{-1} = N(z)\left(I + \mathcal{O}\left(\frac{1}{s}\right)\right)N(z) = I + \mathcal{O}\left(\frac{1}{s}\right) \qquad (3.3.30)$$

We can also quite easily deduce that

$$R(z) = I + \mathcal{O}\left(\frac{1}{s}\right) \tag{3.3.31}$$

for $s \to \infty$:
Note that

$$R(z) = C(z)P(z)^{-1}$$

for $z \in \partial U_\delta$. But we saw in section 3.3.7 that for $z \in U_\delta$ it followed that $P(z) = C(z)$, so for $z \in U_\delta$ we have that $R(z) = I$.
For $z \notin \partial U_\delta$ we have that

$$R(z) = C(z)N(z)^{-1}$$

Observe that in section 3.3.7, the same reasoning would hold for any value of δ, so not just for small values δ. So by (3.3.28) and (3.3.30)

$$R(z) = I + \mathcal{O}\left(\frac{1}{s}\right)$$

3.3.9 The proof of Theorem 3.1.1

With the Deift-Zhou steepest descent analysis complete, we have everything we need to prove our first main theorem:
Lemma 3.3.1 allows us to write

$$\mathbb{J}_\alpha(x,y) = \frac{1}{2\pi i(x-y)}\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right)\widehat{B_{\alpha+}}^{-1}(y)\widehat{B_{\alpha+}}(x)\begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i}\end{pmatrix} \tag{3.3.32}$$

for $x > 0$, $y > 0$.
Equation (3.3.32) can be reformulated, using (3.3.23), as

$$\begin{aligned} \mathbb{J}_\alpha(x+s,y+s) &= \frac{1}{2\pi i(x-y)}\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right)\widehat{B_{\alpha+}}^{-1}(y+s) \\ &\qquad \cdot \widehat{B_{\alpha+}}(x+s)\begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i}\end{pmatrix} \\ &= \frac{1}{2\pi i(x-y)}\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right)A_+^{-1}(y)A_+(x)\begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i}\end{pmatrix} \end{aligned} \tag{3.3.33}$$

for $x > -s$, $y > -s$.
Replacing x with sx and y with sy in (3.3.33), gives because of (3.3.24),

$$\mathbb{J}_\alpha(s(x+1),s(y+1)) = \frac{1}{2\pi is(x-y)}\left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right)A_+^{-1}(sy)A_+(sx)\begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i}\end{pmatrix}$$

for $x > -1$, $y > -1$, or

$$s\mathbb{J}_\alpha(s(x+1), s(y+1)) = \frac{1}{2\pi i(x-y)} \left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) B_+^{-1}(y)B_+(x) \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix}$$

for $x > -1$, $y > -1$.
Remembering that $C(z) = B(z)e^{-(-s(z+1))^{\frac{1}{2}}\sigma_3}$ from (3.3.26), we find that

$$s\mathbb{J}_\alpha(s(x+1), s(y+1)) = \frac{1}{2\pi i(x-y)} \left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) e^{-(-s(y+1))_+^{\frac{1}{2}}\sigma_3} C_+^{-1}(y)$$
$$\cdot C_+(x)e^{(-s(x+1))_+^{\frac{1}{2}}\sigma_3} \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix} \tag{3.3.34}$$

for $x > -1$, $y > -1$.
Now let $x > -1 + \delta$ and $y > -1 + \delta$. Then because of (3.3.29) we see that

$$C_+^{-1}(y)C_+(x) = N_+^{-1}(y)R^{-1}(y)R(x)N_+(x) \tag{3.3.35}$$

So what can be said about $R^{-1}(y)R(x)$? Recall from (3.3.31) that

$$R(z) = I + \mathcal{O}\left(\frac{1}{s}\right)$$

Thus, using Cauchy's theorem, for some closed curve Γ around z, gives

$$\frac{d}{dz}R(z) = \oint_\Gamma \frac{R(t) - I}{(t-z)^2} dt = \mathcal{O}\left(\frac{1}{s}\right)$$

Integrating $\frac{d}{dz}R(z)$ from y to x then gives

$$R(x) - R(y) = \mathcal{O}\left(\frac{x-y}{s}\right)$$

so

$$R^{-1}(y)R(x) = R^{-1}(y)\left(R(y) + \mathcal{O}\left(\frac{x-y}{s}\right)\right) = I + \mathcal{O}\left(\frac{x-y}{s}\right)$$

As $x > -1 + \delta$ and $y > -1 + \delta$, we can expand $N(x)$ as a Taylor series around $x = y$, giving

$$N(x) = N(y) + \mathcal{O}(x-y)$$

Therefore, we can conclude that

$$N_+^{-1}(y)R^{-1}(y)R(x)N_+(x) = I + \mathcal{O}(x-y) \tag{3.3.36}$$

as $y \to x$.
Thus, combining (3.3.34) with (3.3.35) and (3.3.36), we see that

$$s\mathbb{J}_\alpha(s(x+1), s(y+1)) = \frac{1}{2\pi i(x-y)} \left(-e^{-\frac{1}{2}\alpha\pi i}, e^{\frac{1}{2}\alpha\pi i}\right) e^{-(-s(y+1))_+^{\frac{1}{2}}\sigma_3} \tag{3.3.37}$$
$$\cdot (I + \mathcal{O}(x-y))$$
$$\cdot e^{(-s(x+1))_+^{\frac{1}{2}}\sigma_3} \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} \\ e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix}$$

as $y \to x$.
Equation (3.3.37) can in turn be written as

$$\begin{aligned} &s\mathbb{J}_\alpha(s(x+1), s(y+1)) \\ &= \frac{1}{2\pi i(x-y)} \left(e^{-\left((-s(x+1))_+^{\frac{1}{2}} - (-s(y+1))_+^{\frac{1}{2}}\right)} - e^{(-s(x+1))_-^{\frac{1}{2}} - (-s(y+1))_+^{\frac{1}{2}}} \right) + \mathcal{O}(1) \\ &= \frac{1}{2\pi i(x-y)} \left(e^{is^{\frac{1}{2}}\left((x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}}\right)} - e^{-is^{\frac{1}{2}}\left((x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}}\right)} \right) + \mathcal{O}(1) \\ &= \frac{1}{\pi(x-y)} \sin\left(s^{\frac{1}{2}}\left((x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}}\right)\right) + \mathcal{O}(1) \end{aligned} \tag{3.3.38}$$

Obviously, we need to manipulate $s^{\frac{1}{2}}\left((x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}}\right)$ in some way. Observe that by the mean value theorem,

$$(x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}} = (x-y)\frac{1}{2}(1+\xi)^{-\frac{1}{2}}$$

for some ξ between x and y. Substitute $\frac{2\pi x}{s^{\frac{1}{2}}}$ for x and $\frac{2\pi y}{s^{\frac{1}{2}}}$ for y to obtain

$$s^{\frac{1}{2}}\left(\left(\frac{2\pi x}{s^{\frac{1}{2}}}+1\right)^{\frac{1}{2}} - \left(\frac{2\pi y}{s^{\frac{1}{2}}}+1\right)^{\frac{1}{2}}\right) = \pi(x-y)\left(1+\mathcal{O}\left(\frac{1}{s^{\frac{1}{2}}}\right)\right)$$

as $s \to \infty$.
Performing the same substitution for

$$s\mathbb{J}_\alpha(s(x+1), s(y+1)) = \frac{1}{\pi(x-y)} \sin\left(s^{\frac{1}{2}}\left((x+1)^{\frac{1}{2}} - (y+1)^{\frac{1}{2}}\right)\right) + \mathcal{O}(1)$$

we may thus deduce that

$$s\mathbb{J}_\alpha\left(s\left(1+\frac{2\pi x}{s^{\frac{1}{2}}}\right), s\left(1+\frac{2\pi y}{s^{\frac{1}{2}}}\right)\right) = \frac{s^{\frac{1}{2}}}{2\pi}\frac{\sin\pi(x-y)}{\pi(x-y)} + \mathcal{O}(1)$$

and therefore

$$2\pi s^{\frac{1}{2}}\mathbb{J}_\alpha\left(s+2\pi x s^{\frac{1}{2}}, s+2\pi y s^{\frac{1}{2}}\right)=\frac{\sin\pi(x-y)}{\pi(x-y)}+\mathcal{O}\left(\frac{1}{s^{\frac{1}{2}}}\right)$$

or, equivalently, by replacing $s^{\frac{1}{2}}$ by s

$$2\pi s\mathbb{J}_\alpha\left(s^2+2\pi x s, s^2+2\pi y s\right)=\frac{\sin\pi(x-y)}{\pi(x-y)}+\mathcal{O}\left(\frac{1}{s}\right)$$

as $s\to\infty$.

3.4 Proof of Theorem 3.1.2

3.4.1 Relating $\mathbb{K}_c^{CHF}$ to the Confluent Hypergeometric Riemann-Hilbert Problem

First we will recall the Riemann-Hilbert problem for confluent hypergeometric functions.
Let $c>0$, $c\neq 1$ and $\lambda=\frac{i}{\pi}\log c$. We consider a matrix valued function $C_c:\mathbb{C}\setminus\Gamma_C\to\mathbb{C}^{2\times 2}$ (see Figure 3.5) that satisfies the following conditions:

- $C_c(z)$ is analytic for $z\in\mathbb{C}\setminus\Gamma_C$ where $\Gamma_C=\bigcup_{i=1}^{6}\Gamma_{iC}$ (see Figure 3.5).
-
$$C_{c+}(z)=C_{c-}(z)\begin{pmatrix}0 & c\\ -c^{-1} & 0\end{pmatrix}\text{ for } z\in\Gamma_{4C} \tag{3.4.1}$$
$$C_{c+}(z)=C_{c-}(z)\begin{pmatrix}0 & c^{-1}\\ -c & 0\end{pmatrix}\text{ for } z\in\Gamma_{3C} \tag{3.4.2}$$
$$C_{c+}(z)=C_{c-}(z)\begin{pmatrix}1 & 0\\ c^{-1} & 1\end{pmatrix}\text{ for } z\in\Gamma_{2C}\cup\Gamma_{6C} \tag{3.4.3}$$
$$C_{c+}(z)=C_{c-}(z)\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\text{ for } z\in\Gamma_{1C}\cup\Gamma_{5C} \tag{3.4.4}$$
- For $z\to\infty$
$$C_c(z)=\left(I+\mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}c^{\sigma_3}e^{-iz\sigma_3}\text{ for Im } z>0 \tag{3.4.5}$$
$$C_c(z)=\left(I+\mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}e^{iz\sigma_3}\text{ for Im } z\leq 0 \tag{3.4.6}$$
where the cut of z^λ is chosen along the negative real axis.
- For $z\notin II\cup V$ and close to 0
$$C_c(z)=\mathcal{O}(\log|z|)$$

- For $z \in II \cup V$ and close to 0

$$C_c(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z| \\ 1 & \log|z| \end{pmatrix}$$

which has a solution

$$C_c(z) = \begin{cases} \begin{pmatrix} c^{-1}\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \\ -c^{-1}\frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) & \psi(-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in I \\ \begin{pmatrix} \Gamma(1-\lambda)\phi(\lambda,1;2iz) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \\ \Gamma(1+\lambda)\phi(1+\lambda,1;2iz) & \psi(-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in II \\ \begin{pmatrix} c\psi(\lambda,1;2e^{-\frac{3}{2}\pi i}z) & -\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \\ -c\frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{-\frac{3}{2}\pi i}z) & \psi(-\lambda,1;2e^{-\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in III \\ \begin{pmatrix} c\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{\frac{3}{2}\pi i}z) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ -c\psi(-\lambda,1;2e^{\frac{3}{2}\pi i}z) & \frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in IV \\ \begin{pmatrix} \Gamma(1-\lambda)\phi(\lambda,1;2iz) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ \Gamma(1+\lambda)\phi(1+\lambda,1;2iz) & \frac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in V \\ \begin{pmatrix} c^{-1}\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)}\psi(1-\lambda,1;2e^{-\frac{1}{2}\pi i}z) & -\psi(\lambda,1;2e^{\frac{1}{2}\pi i}z) \\ -c^{-1}\psi(-\lambda,1;2e^{-\frac{1}{2}\pi i}z) & \frac{\Gamma(1+\lambda)}{\Gamma(-\lambda)}\psi(1+\lambda,1;2e^{\frac{1}{2}\pi i}z) \end{pmatrix} e^{-iz\sigma_3} \\ \qquad \text{for } z \in VI \end{cases} \tag{3.4.7}$$

As we did for the Bessel kernel, first we will link the confluent hypergeometric kernel to the Riemann-Hilbert problem for confluent hypergeometric functions. For this purpose, we will define for $x_0 \in \mathbb{R}$ fixed, the function

$$\nu_0(x) = \begin{cases} 1 \text{ if } x < 0 \\ c^2 \text{ if } x \geq 0 \end{cases}$$

Note that

$$\begin{aligned} \frac{\nu_0(x)}{c} &= \begin{cases} \frac{1}{c} \text{ if } x < 0 \\ c \text{ if } x \geq 0 \end{cases} \\ &= c^{\operatorname{sgn}(x)} \end{aligned}$$

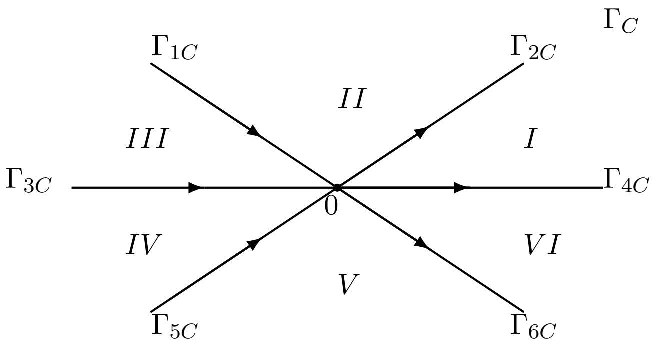

Figure 3.5: Sketch of curves on which C_c has jumps

Lemma 3.4.1. For x, $y \in \mathbb{R}$

$$\mathbb{K}_c^{CHF}(x,y) = \frac{1}{2\pi i(x-y)} \left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(\pi y)^{-1} \cdot C_{c+}(\pi x) \begin{pmatrix} c^{\frac{1}{2}\operatorname{sgn}(x)} \\ c^{-\frac{1}{2}\operatorname{sgn}(x)} \end{pmatrix}$$

Proof. First we should check whether C_c is invertible.
Note that all jump matrices have determinant equal to 1 (see (3.4.1)-(3.4.4)), so just as was found in the proof of Lemma 3.3.1, we see that

$$\det C_{c+}(z) = \det C_{c-}(z)$$

for $z \in \Gamma_C$, meaning that C_c is analytic away from 0.
For $z \to 0$, we get that

$$\det C_c(z) = \mathcal{O}(\log |z|)$$

so around 0 the singularity is removable.
For $z \to \infty$, $\det C_c(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right)$, so by Liouville's theorem, $\det C_c(z) = 1$, meaning that C_c is invertible.
Next, observe that for $y < 0$

$$\begin{aligned} \left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(y)^{-1} &= \left(-c^{\frac{1}{2}}, c^{-\frac{1}{2}}\right) C_{c+}(y)^{-1} \\ &= \left(0, c^{-\frac{1}{2}}\right) \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} C_{c+}(y)^{-1} \end{aligned} \tag{3.4.8}$$

Note that for $y < 0$, we have that

$$C_{c+}(y) \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

is the analytic continuation of $C_c(z)$, $z \in II$ to the negative real axis (see Figure 3.5 and (3.4.4)).
Thus, writing out (3.4.8), using (3.4.7) in II, we get that

$$\begin{aligned}
&\left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(y)^{-1} = \left(0, c^{-\frac{1}{2}}\right)\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} C_{c+}(y)^{-1} \\
&= c^{-\frac{1}{2}}\left(-\Gamma(1+\lambda)\phi(1+\lambda,1;2iy)e^{-iy}, \Gamma(1-\lambda)\phi(\lambda,1;2iy)e^{-iy}\right) \qquad (3.4.9)
\end{aligned}$$

Analogously, we find that for $y > 0$

$$\begin{aligned}
&\left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(y)^{-1} \\
&= c^{\frac{1}{2}}\left(-\Gamma(1+\lambda)\phi(1+\lambda,1;2iy)e^{-iy}, \Gamma(1-\lambda)\phi(\lambda,1;2iy)e^{-iy}\right) \qquad (3.4.10)
\end{aligned}$$

Equations (3.4.9) and (3.4.10) can be combined into

$$\begin{aligned}
&\left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(y)^{-1} \\
&= c^{\frac{1}{2}\operatorname{sgn}(y)}\left(-\Gamma(1+\lambda)\phi(1+\lambda,1;2iy)e^{-iy}, \Gamma(1-\lambda)\phi(\lambda,1;2iy)e^{-iy}\right) \qquad (3.4.11)
\end{aligned}$$

In the same way, we can deduce that

$$C_{c+}(x)\begin{pmatrix} c^{\frac{1}{2}\operatorname{sgn}(x)} \\ c^{-\frac{1}{2}\operatorname{sgn}(x)} \end{pmatrix} = c^{\frac{1}{2}\operatorname{sgn}(x)}\begin{pmatrix} \Gamma(1-\lambda)\phi(\lambda,1;2ix)e^{-ix} \\ \Gamma(1+\lambda)\phi(1+\lambda,1;2ix)e^{-ix} \end{pmatrix} \qquad (3.4.12)$$

Combining (3.4.11) and (3.4.12) and inserting πx for x and πy for y, we get

$$\begin{aligned}
&\left(-c^{-\frac{1}{2}\operatorname{sgn}(y)}, c^{\frac{1}{2}\operatorname{sgn}(y)}\right) C_{c+}(\pi y)^{-1} C_{c+}(\pi x)\begin{pmatrix} c^{\frac{1}{2}\operatorname{sgn}(x)} \\ c^{-\frac{1}{2}\operatorname{sgn}(x)} \end{pmatrix} \\
&= \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\Gamma(1+\lambda)\Gamma(1-\lambda)}{c}[G(1+\lambda, 2\pi ix), G(\lambda, 2\pi iy)] \qquad (3.4.13)
\end{aligned}$$

Note that if we multiply both sides of (3.4.13) with

$$\frac{c\log c}{\pi i(x-y)(c^2-1)\Gamma(1+\lambda)\Gamma(1-\lambda)}$$

we get exactly the Confluent Hypergeometric kernel on the left hand side. If

$$\frac{c\log c}{\pi i(c^2-1)\Gamma(1+\lambda)\Gamma(1-\lambda)} = -\frac{c\left(c^{-1}-c\right)}{2\pi i(c^2-1)} = \frac{1}{2\pi i}$$

then we have proven our lemma.
Observe that

$$\frac{c\log c}{\pi i(c^2-1)\Gamma(1+\lambda)\Gamma(1-\lambda)} = \frac{c}{(c^2-1)\lambda\Gamma(\lambda)\Gamma(-\lambda)} \qquad (3.4.14)$$

where we have used that $\lambda = \frac{i \log c}{\pi}$ and $\Gamma(1 \pm \lambda) = \pm\lambda\Gamma(\pm\lambda)$.
Making use of the identity

$$\Gamma(\lambda)\Gamma(-\lambda) = \frac{-\pi}{\lambda \sin(\pi\lambda)}$$

we write (3.4.14) as

$$\frac{c \log c}{\pi i(c^2-1)\Gamma(1+\lambda)\Gamma(1-\lambda)} = -\frac{c \sin(\pi\lambda)}{\pi(c^2-1)} = -\frac{c\left(e^{\pi i\lambda} - e^{-\pi i\lambda}\right)}{2\pi i(c^2-1)} \tag{3.4.15}$$

Using once more that $\lambda = \frac{i \log c}{\pi}$, (3.4.15) becomes

$$\frac{c \log c}{\pi i(c^2-1)\Gamma(1+\lambda)\Gamma(1-\lambda)} = -\frac{c\left(c^{-1} - c\right)}{2\pi i(c^2-1)} = \frac{1}{2\pi i}$$

which completes the proof. □

3.4.2 The Deift-Zhou method of steepest descent

Our goal is to transform the Riemann Hilbert Problem as formulated in section 3.4.1 using Deift Zhou steepest descent analysis, following essentially the same steps as we did for the case of the Bessel Riemann Hilbert Problem. In the following analysis, we will focus on the case that $s \to \infty$. The case that $s \to -\infty$ will be obvious after completing the analysis for $s \to \infty$.

The First Step: $C_c \mapsto A$

Define for $s > 0$ that

$$A(z) = s^{\lambda\sigma_3} C_c(s(z+1)) \tag{3.4.16}$$

Furthermore, let $\Gamma_A = \bigcup_{i=1}^{6} \Gamma_{jA}$ (see Figure 3.6) be Γ_C (see Figure 3.5) shifted horizontally to the left from 0 to -1.
Then $A : \mathbb{C} \setminus \Gamma_A \to \mathbb{C}^{2\times 2}$ satisfies the following conditions:

- $A(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_A$ (see Figure 3.6).
- $$A_+(z) = A_-(z) \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \text{ for } z \in \Gamma_{4A}$$
$$A_+(z) = A_-(z) \begin{pmatrix} 0 & c^{-1} \\ -c & 0 \end{pmatrix} \text{ for } z \in \Gamma_{3A}$$
$$A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{2A} \cup \Gamma_{6A}$$
$$A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ for } z \in \Gamma_{1A} \cup \Gamma_{5A}$$

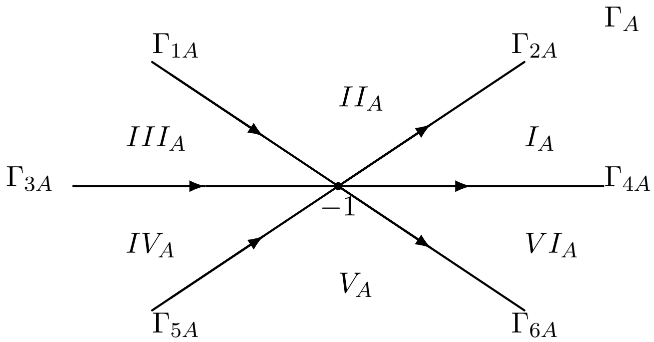

Figure 3.6: Sketch of curves on which A has jumps

- For $z \to \infty$

$$A(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}c^{\sigma_3}e^{-is(z+1)\sigma_3} \text{ for Im } z > 0 \quad (3.4.17)$$

$$A(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}e^{is(z+1)\sigma_3} \text{ for Im } z \leq 0 \quad (3.4.18)$$

where the cut of z^λ is chosen along the negative real axis.

- For $z \notin II_A \cup V_A$ and close to -1

$$A(z) = \mathcal{O}(\log|z+1|)$$

- For $z \in II_A \cup V_A$ and close to 0

$$A(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}$$

The Second Step: $A \mapsto B$

Define for $s > 0$

$$B(z) = A(z)\begin{cases} e^{is(z+1)\sigma_3} & \text{for Im } z > 0 \\ e^{-is(z+1)\sigma_3} & \text{for Im } z < 0 \end{cases} \quad (3.4.19)$$

Then

- $B(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_A$ (see Figure 3.6).

- $$\begin{aligned}
B_+(z) &= B_-(z)\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \text{ for } z \in \Gamma_{4A} \\
B_+(z) &= B_-(z)\begin{pmatrix} 0 & c^{-1} \\ -c & 0 \end{pmatrix} \text{ for } z \in \Gamma_{3A} \\
B_+(z) &= B_-(z)\begin{pmatrix} 1 & 0 \\ c^{-1}e^{2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{2A} \\
B_+(z) &= B_-(z)\begin{pmatrix} 1 & 0 \\ c^{-1}e^{-2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{6A} \\
B_+(z) &= B_-(z)\begin{pmatrix} 1 & 0 \\ ce^{2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{1A} \\
B_+(z) &= B_-(z)\begin{pmatrix} 1 & 0 \\ ce^{-2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{5A}
\end{aligned}$$

- For $z \to \infty$

$$B(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}c^{\sigma_3} \text{ for Im } z > 0 \tag{3.4.20}$$

$$B(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for Im } z \leq 0 \tag{3.4.21}$$

  where the cut of z^λ is chosen along the negative real axis.

- For $z \notin II_A \cup V_A$ and close to -1

$$B(z) = \mathcal{O}\left(\log|z+1|\right)$$

- For $z \in II_A \cup V_A$ and close to -1

$$B(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}$$

Note that for Im $z > 0$, Re $is(z+1) < 0$ and for Im $z < 0$, Re $is(z+1) > 0$, so the jumps on Γ_{1A}, Γ_{2A}, Γ_{5A} and Γ_{6A} go to the identity matrix for $s \to \infty$, which leads us to the parametrix away from $z = -1$ in the next section.

The Parametrix away from $z = -1$

The parametrix away from the point $z = -1$ can be described as a solution N of the problem

- $N(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$

- N has the following jump conditions:

$$N_+(z) = \begin{cases} N_-(z) \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} & \text{for } z > -1 \\ N_-(z) \begin{pmatrix} 0 & c^{-1} \\ -c & 0 \end{pmatrix} & \text{for } z < -1 \end{cases}$$

- For $z \to \infty$

$$N(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3} c^{\sigma_3} \text{ for Im } z > 0 \tag{3.4.22}$$

$$N(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)(2z)^{-\lambda\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for Im } z < 0 \tag{3.4.23}$$

  where the cut of z^λ is chosen along the negative real axis.

A solution to this problem is

$$N(z) = (2(z+1))^{-\lambda\sigma_3} c^{\sigma_3} \text{ for Im } z > 0$$
$$N(z) = (2(z+1))^{-\lambda\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for Im } z < 0$$

The Parametrix near $z = -1$

Within a disk U_δ of radius $0 < \delta << 1$, centered around $z = -1$, we create a parametrix P fulfilling the following conditions:

- $P(z)$ is analytic for $z \in U_\delta \setminus \Gamma_A$ (see Figure 3.6)

- $$P_+(z) = P_-(z) \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \text{ for } z \in \Gamma_{4A}$$
  $$P_+(z) = P_-(z) \begin{pmatrix} 0 & c^{-1} \\ -c & 0 \end{pmatrix} \text{ for } z \in \Gamma_{3A}$$
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ c^{-1}e^{2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{2A}$$
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ c^{-1}e^{-2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{6A}$$
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ ce^{2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{1A}$$
  $$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ ce^{-2is(z+1)} & 1 \end{pmatrix} \text{ for } z \in \Gamma_{5A}$$

- For $s \to \infty$

$$P(z) = \left(I + \mathcal{O}\left(\frac{1}{s}\right)\right) N(z) \text{ uniformly for } z \in \partial U_\delta \tag{3.4.24}$$

- For $z \notin II_A \cup V_A$ and close to -1

$$P(z) = \mathcal{O}\left(\log|z+1|\right)$$

- For $z \in II_A \cup V_A$ and close to -1

$$P(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}$$

As for the Bessel case, finding P is simplicity itself, as B fulfills all conditions stated above, so $P = B$.
As was mentioned in the Bessel case, here as well it should be stated that P is not really a parametrix, as it is not approximating B, it in fact is B.

The Third Step: $B \mapsto R$

Define

$$R(z) = \begin{cases} B(z)P(z)^{-1} \text{ if } z \in U_\delta \\ B(z)N(z)^{-1} \text{ if } z \notin U_\delta \end{cases} \tag{3.4.25}$$

Let $\Gamma_{1R} = \Gamma_{2A} \cap U_\delta^c$, $\Gamma_{2R} = \Gamma_{1A} \cap U_\delta^c$, $\Gamma_{3R} = \Gamma_{5A} \cap U_\delta^c$ and $\Gamma_{4R} = \Gamma_{6A} \cap U_\delta^c$.
Furthermore, let

$$\Gamma_R = \bigcup_{i=1}^{4} \Gamma_{iA} \cup \partial U_\delta$$

Then we find

- $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_R$ (see Figure 3.7)
- R has the following jump conditions:

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ c^{-1}e^{2is(z+1)} & 1 \end{pmatrix} N(z)^{-1} \text{ for } z \in \Gamma_{1R}$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ ce^{2is(z+1)} & 1 \end{pmatrix} N(z)^{-1} \text{ for } z \in \Gamma_{2R}$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ ce^{-2is(z+1)} & 1 \end{pmatrix} N(z)^{-1} \text{ for } z \in \Gamma_{3R}$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ c^{-1}e^{-2is(z+1)} & 1 \end{pmatrix} N(z)^{-1} \text{ for } z \in \Gamma_{4R}$$

$$R_+(z) = R_-(z)P(z)N(z)^{-1} \text{ for } z \in \partial U_\delta$$

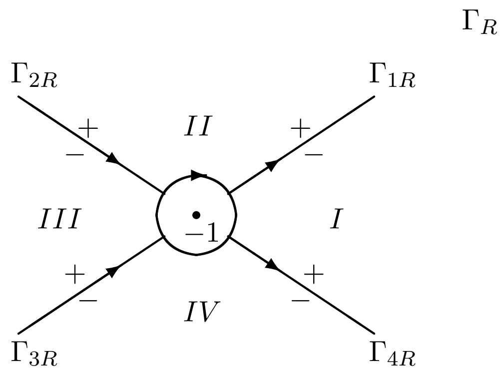

Figure 3.7: Sketch of curves on which R has jumps

- For $z \to \infty$

$$R(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$$

Thus we complete our Deift-Zhou steepest descent analysis.
Note that on $\bigcup_{i=1}^{4} \Gamma_{iR}$ the jumps go to the identity matrix for the same reason that the jumps of B on $\Gamma_{1A} \cup \Gamma_{2A} \cup \Gamma_{5A} \cup \Gamma_{6A}$ go to the identity matrix.
For $z \in \partial U_\delta$ we have that because of (3.4.24)

$$P(z)N(z)^{-1} = \left(I + \mathcal{O}\left(\frac{1}{s}\right)\right) N(z)N(z)^{-1} = I + \mathcal{O}\left(\frac{1}{s}\right)$$

for $s \to \infty$, which proves that the jump of R on ∂U_δ goes to the identity matrix as well.
By the same reasoning as for (3.3.31) we also find that

$$R(z) = I + \mathcal{O}\left(\frac{1}{s}\right) \tag{3.4.26}$$

3.4.3 Proof of Theorem 3.1.2

As for the Bessel case, we will go through the Deift-Zhou steepest descent analysis for the Riemann Hilbert problem, linking the Confluent Hypergeometric kernel to the sine kernel.
Recall from (3.4.16)that the first transformation was given by

$$A(z) = s^{\lambda\sigma_3} C_c(s(z+1))$$

By Lemma 3.4.1, for $x > 0$, $y > 0$

$$\frac{1}{\pi}\mathbb{K}_c^{CHF}\left(\frac{x}{\pi}, \frac{y}{\pi}\right) = \frac{1}{2\pi i(x-y)}\left(-\left(\frac{\nu_0(y)}{c}\right)^{-\frac{1}{2}}, \left(\frac{\nu_0(y)}{c}\right)^{\frac{1}{2}}\right) C_{c+}(y)^{-1} \cdot C_{c+}(x)\begin{pmatrix}\left(\frac{\nu_0(x)}{c}\right)^{\frac{1}{2}} \\ \left(\frac{\nu_0(x)}{c}\right)^{-\frac{1}{2}}\end{pmatrix}$$
$$= \frac{1}{2\pi i(x-y)}\left(-c^{-\frac{1}{2}}, c^{\frac{1}{2}}\right) C_{c+}(y)^{-1}C_{c+}(x)\begin{pmatrix} c^{\frac{1}{2}} \\ c^{-\frac{1}{2}}\end{pmatrix} \tag{3.4.27}$$

can be rewritten as

$$\frac{s}{\pi}\mathbb{K}_c^{CHF}\left(\frac{s}{\pi}(x+1), \frac{s}{\pi}(y+1)\right) = \frac{1}{2\pi i(x-y)}(-c^{-\frac{1}{2}}, c^{\frac{1}{2}})A_+(y)^{-1}A_+(x)\begin{pmatrix} c^{\frac{1}{2}} \\ c^{-\frac{1}{2}}\end{pmatrix}$$

for $x > -1$, $y > -1$ and $s > 0$.
The second transformation that was performed in our steepest descent analysis was (see (3.4.19))

$$B(z) = A(z)\begin{cases} e^{is(z+1)\sigma_3} & \text{for Im } z > 0 \\ e^{-is(z+1)\sigma_3} & \text{for Im } z < 0 \end{cases}$$

Thus, we can deduce the following expression:

$$\frac{s}{\pi}\mathbb{K}_c^{CHF}\left(\frac{s}{\pi}(x+1), \frac{s}{\pi}(y+1)\right) = \frac{1}{2\pi i(x-y)}(-c^{-\frac{1}{2}}, c^{\frac{1}{2}})e^{is(y+1)\sigma_3}B_+(y)^{-1} \cdot B_+(x)e^{-is(x+1)\sigma_3}\begin{pmatrix} c^{\frac{1}{2}} \\ c^{-\frac{1}{2}}\end{pmatrix} \tag{3.4.28}$$

for $x > -1$, $y > -1$.
As before for the proof of Theorem 3.1.1, choosing $x > -1+\delta$, $y > -1+\delta$ and using that $B(z) = R(z)N(z)$ accordingly, we find that

$$B_+(y)^{-1}B_+(x) = N_+(y)^{-1}R(y)^{-1}R(x)N_+(x) = I + \mathcal{O}(x-y) \tag{3.4.29}$$

as $y \to x$.
Inserting (3.4.29) into (3.4.28) gives

$$\begin{aligned}
&\frac{s}{\pi}\mathbb{K}_c^{CHF}\left(\frac{s}{\pi}(x+1), \frac{s}{\pi}(y+1)\right) \\
&= \frac{1}{2\pi i(x-y)}(-c^{-\frac{1}{2}}, c^{\frac{1}{2}})e^{is(y+1)\sigma_3} I e^{-is(x+1)\sigma_3}\begin{pmatrix} c^{\frac{1}{2}} \\ c^{-\frac{1}{2}} \end{pmatrix} + \mathcal{O}(1) \\
&= \frac{1}{2\pi i(x-y)}(-c^{-\frac{1}{2}}, c^{\frac{1}{2}})e^{is(y-x)\sigma_3}\begin{pmatrix} c^{\frac{1}{2}} \\ c^{-\frac{1}{2}} \end{pmatrix} + \mathcal{O}(1) \\
&= \frac{e^{is(x-y)} - e^{-is(x-y)}}{2\pi i(x-y)} + \mathcal{O}(1) \\
&= \frac{\sin\left(s(x-y)\right)}{\pi(x-y)} + \mathcal{O}(1)
\end{aligned}$$

for $x > -1+\delta$, $y > -1+\delta$.
Replacing x by $\frac{\pi x}{s}$ and y by $\frac{\pi y}{s}$ will then finally reveal that

$$\mathbb{K}^{CHF}\left(x + \frac{s}{\pi}, y + \frac{s}{\pi}\right) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

for $x > -s$, $y > -s$ for $s \to \infty$.
Using the same strategy except for translating to the right instead of to the left in the previous steepest descent analysis, one finds that for $s > 0$

$$\mathbb{K}_c^{CHF}\left(x - \frac{s}{\pi}, y - \frac{s}{\pi}\right) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

for $x < s$, $y < s$ for $s \to \infty$. This means that for x, $y \in \mathbb{R}$

$$\mathbb{K}_c^{CHF}\left(x + s, y + s\right) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

for $s \to \pm\infty$, which completes the proof.

3.5 Proof of Theorem 3.1.3

3.5.1 Two lemmas

In order to prove Theorem 3.1.3, we will first prove Lemma 3.5.1 and Lemma 3.5.2, as can be seen below. The identities explained in Lemma 3.5.1 are taken from [2] and Lemma 3.5.2 gives the main identity used in the proof of Theorem 3.1.3.

Lemma 3.5.1.

$$\alpha J_\alpha(z) + zJ'_\alpha(z) = zJ_{\alpha-1}(z) \tag{3.5.1}$$

and

$$-\alpha J_\alpha(z) + zJ'_\alpha(z) = -zJ_{\alpha+1}(z) \tag{3.5.2}$$

Proof. Observe that

$$\begin{aligned}
\frac{d}{dz}\left(z^\alpha J_\alpha(z)\right) &= \frac{d}{dz}\left(z^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{2m+\alpha}}{m!\Gamma(m+\alpha+1)}\right) \\
&= \frac{d}{dz}\left(\sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\right)^{2m+\alpha} (z)^{2m+2\alpha}}{m!\Gamma(m+\alpha+1)}\right) \\
&= z^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\alpha) \left(\frac{1}{2}\right)^{2m+\alpha} \left(\frac{1}{2}z\right)^{2m+\alpha-1}}{m!(m+\alpha)\Gamma(m+\alpha)} \\
&= z^\alpha J'_{\alpha-1}(z)
\end{aligned}$$

Using the product rule on the left hand side of $\frac{d}{dz}\left(z^\alpha J_\alpha(z)\right) = z^\alpha J'_{\alpha-1}(z)$ and then dividing by $z^{\alpha-1}$ gives (3.5.1). (3.5.2) can be deduced using a similar argument. □

Lemma 3.5.2. Let u, $v \in \mathbb{R}$. Furthermore, let for $u < 0$ $u^\alpha = e^{\alpha\pi i}|u|^\alpha$ and $\sqrt{u} = e^{\frac{1}{2}\pi i}|u|^{\frac{1}{2}}$. Then

$$\mathbb{J}^0_\alpha(u,v) = \pi^2 \left(\frac{|u|}{u}\right)^\alpha \left(\frac{|v|}{v}\right)^\alpha \sqrt{u}\sqrt{v}\left(\mathbb{J}_{\alpha-\frac{1}{2}}(\pi^2u^2, \pi^2v^2) + \mathbb{J}_{\alpha+\frac{1}{2}}(\pi^2u^2, \pi^2v^2)\right) \tag{3.5.3}$$

Proof.

$$\begin{aligned}
\mathbb{J}_{\alpha-\frac{1}{2}}(u^2,v^2) + \mathbb{J}_{\alpha+\frac{1}{2}}(u^2,v^2) &= \frac{J_{\alpha-\frac{1}{2}}(u)vJ'_{\alpha-\frac{1}{2}}(v) - uJ'_{\alpha-\frac{1}{2}}(u)J_{\alpha-\frac{1}{2}}(v)}{2(u^2-v^2)} \\
&+ \frac{J_{\alpha+\frac{1}{2}}(u)vJ'_{\alpha+\frac{1}{2}}(v) - uJ'_{\alpha+\frac{1}{2}}(u)J_{\alpha+\frac{1}{2}}(v)}{2(u^2-v^2)}
\end{aligned}$$

Using (3.5.1) and (3.5.2) to rewrite $vJ'_{\alpha+\frac{1}{2}}(v)$ and $uJ'_{\alpha+\frac{1}{2}}(u)$ then gives

$$\begin{aligned}
\mathbb{J}_{\alpha-\frac{1}{2}}(u^2,v^2) + \mathbb{J}_{\alpha+\frac{1}{2}}(u^2,v^2) &= \frac{(u+v)\left(J_{\alpha+\frac{1}{2}}(u)J_{\alpha-\frac{1}{2}}(v) - J_{\alpha-\frac{1}{2}}(u)J_{\alpha+\frac{1}{2}}(v)\right)}{2(u^2-v^2)} \\
&= \frac{\left(J_{\alpha+\frac{1}{2}}(u)J_{\alpha-\frac{1}{2}}(v) - J_{\alpha-\frac{1}{2}}(u)J_{\alpha+\frac{1}{2}}(v)\right)}{2(u-v)}
\end{aligned}$$

So replacing u for πu and v for πv and then multiplying everything by $\pi^2\left(\frac{|u|}{u}\right)^\alpha\left(\frac{|v|}{v}\right)^\alpha\sqrt{u}\sqrt{v}$ gives (3.5.3). □

This concludes our preparation for proving the third main theorem.

3.5.2 Proof of Theorem 3.1.3

Remember from Theorem 3.1.1 that

$$2\pi s^{\frac{1}{2}}\mathbb{J}_\alpha\left(s+2\pi x s^{\frac{1}{2}}, s+2\pi y s^{\frac{1}{2}}\right) = \frac{\sin\pi(x-y)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s^{\frac{1}{2}}}\right)$$

and recall from Lemma 3.5.2 that

$$\mathbb{J}^0_\alpha(u,v) = \pi^2\left(\frac{|u|}{u}\right)^\alpha\left(\frac{|v|}{v}\right)^\alpha\sqrt{u}\sqrt{v}\left(\mathbb{J}_{\alpha-\frac{1}{2}}(\pi^2u^2,\pi^2v^2)+\mathbb{J}_{\alpha+\frac{1}{2}}(\pi^2u^2,\pi^2v^2)\right) \tag{3.5.4}$$

We will prove Theorem 3.1.3 for $s\to\infty$ and then remark how the result also holds for $s\to-\infty$.
Note that for

$$u = \frac{s}{\pi}+x \text{ and } v = \frac{s}{\pi}+y \tag{3.5.5}$$

we have that

$$\left(\frac{|u|}{u}\right)^\alpha\left(\frac{|v|}{v}\right)^\alpha\sqrt{u}\sqrt{v} = \frac{s}{\pi}\left(1+\mathcal{O}\left(\frac{1}{s}\right)\right) \tag{3.5.6}$$

for $s\to\infty$.
Combining (3.5.4), (3.5.5) and (3.5.6) then gives

$$\begin{aligned}
&\mathbb{J}^0_\alpha\left(\frac{s}{\pi}+x,\frac{s}{\pi}+y\right)\\
&=\pi^2\left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\left(\mathbb{J}_{\alpha-\frac{1}{2}}((s+\pi x)^2,(s+\pi y)^2)+\mathbb{J}_{\alpha+\frac{1}{2}}((s+\pi x)^2,(s+\pi y)^2)\right)\\
&=\pi s\left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\left(\mathbb{J}_{\alpha-\frac{1}{2}}(s^2+2\pi xs+\pi^2x^2,s^2+2\pi ys+\pi^2y^2)\right.\\
&\qquad\left.+\mathbb{J}_{\alpha+\frac{1}{2}}(s^2+2\pi xs+\pi^2x^2,s^2+2\pi ys+\pi^2y^2)\right)\\
&=\pi s\left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\left(\mathbb{J}_{\alpha-\frac{1}{2}}\left(s^2+2\pi s\left(x+\frac{\pi x^2}{2s}\right),s^2+2\pi s\left(y+\frac{\pi y^2}{2s}\right)\right)\right.\\
&\qquad\left.+\mathbb{J}_{\alpha+\frac{1}{2}}\left(s^2+2\pi s\left(x+\frac{\pi x^2}{2s}\right),s^2+2\pi s\left(y+\frac{\pi y^2}{2s}\right)\right)\right)
\end{aligned}$$

Applying Theorem 3.1.1 whilst replacing x with $x+\frac{\pi x^2}{2s}$ and y with $y+\frac{\pi y^2}{2s}$ in (A.5.1) then reveals that

$$\begin{aligned}
\mathbb{J}^0_\alpha\left(\frac{s}{\pi}+x,\frac{s}{\pi}+y\right) &= \left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\left(\frac{\sin\left(\pi(x-y)+\frac{\pi^2(x^2-y^2)}{2s}\right)}{\pi(x-y)+\frac{\pi^2(x^2-y^2)}{2\pi s}}+\mathcal{O}\left(\frac{1}{s}\right)\right)\\
&= \left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\left(\frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)}\frac{1+\mathcal{O}\left(\frac{1}{s}\right)}{1+\mathcal{O}\left(\frac{1}{s}\right)}+\mathcal{O}\left(\frac{1}{s}\right)\right)\\
&= \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)}+\mathcal{O}\left(\frac{1}{s}\right)
\end{aligned} \tag{3.5.7}$$

for $s \to \infty$.
Note that for the case that $s < 0$ nothing really changes: Let $x,\, y < 0$. Write $x = |x|e^{\pi i}$ and $y = |y|e^{\pi i}$. Recall from (3.1.5) that for $x,\, y \in \mathbb{R}$

$$\mathbb{J}^0_\alpha(x,y) = \pi \left(\frac{|x|}{x}\right)^\alpha \left(\frac{|y|}{y}\right)^\alpha \sqrt{x}\sqrt{y}\frac{J_{\alpha+\frac{1}{2}}(\pi x)J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x)J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)}$$

So for x and y negative

$$\begin{aligned}\mathbb{J}^0_\alpha(x,y) = \frac{\pi e^{2\alpha\pi i}\sqrt{|x|}\sqrt{|y|}e^{\pi i}}{2(x-y)} &\Big(e^{(\alpha+\frac{1}{2})\pi i}J_{\alpha+\frac{1}{2}}(\pi|x|)e^{(\alpha-\frac{1}{2})\pi i}J_{\alpha-\frac{1}{2}}(\pi|y|) \\ &-e^{(\alpha-\frac{1}{2})\pi i}J_{\alpha-\frac{1}{2}}(\pi|x|)e^{(\alpha+\frac{1}{2})\pi i}J_{\alpha+\frac{1}{2}}(\pi|y|)\Big)\end{aligned}$$

(see [2] or (A.3.1) for an expression for $J_{\alpha\pm\frac{1}{2}}$).
So

$$\mathbb{J}^0_\alpha(x,y) = \pi\sqrt{|x|}\sqrt{|y|}\frac{J_{\alpha+\frac{1}{2}}(\pi|x|)J_{\alpha-\frac{1}{2}}(\pi|y|) - J_{\alpha-\frac{1}{2}}(\pi|x|)J_{\alpha+\frac{1}{2}}(\pi|y|)}{2(|x|-|y|)} \tag{3.5.8}$$

Using (3.5.8) with Lemma 3.5.2 we can now proceed with proving that for $s > 0$

$$\mathbb{J}^0_\alpha(x-s, y-s) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

as $s \to \infty$ as we did for the proof of (3.5.7), other than that when we use Theorem 3.1.1 and replace x with $x + \frac{\pi x^2}{2s}$ and y with $y + \frac{\pi y^2}{2s}$, we should replace x and y with $-x - \frac{\pi x^2}{2s}$ and y with $-y - \frac{\pi y^2}{2s}$ instead. This completes the proof.

Chapter 4

The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights

4.1 Introduction

We consider the asymptotic behavior of the recurrence coefficients $a_{n,N}$ and $b_{n,N}$ in the three-term recurrence relation

$$x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$$

for orthogonal polynomials with respect to varying exponential weights[1]. Here $\pi_{n,N}$ is the n-th degree monic orthogonal polynomial with respect to a varying weight

$$w_N(x) = e^{-NV(x)}$$

where V is real analytic on $\mathbb{R}$ with $\lim\limits_{x\to\pm\infty} \frac{V(x)}{\log(1+x^2)} = +\infty$. Moreover, V is assumed to be one-cut regular, which means that the equilibrium measure $d\mu_V = \psi_V(x)dx$ associated with V is supported on one interval $[a,b]$ where it has the form

$$\psi_V(x)\,dx = \sqrt{(b-x)(x-a)}h(x)\chi_{[a,b]}(x)\,dx \tag{4.1.1}$$

where h is real analytic, strictly positive on $[a,b]$, and in addition the inequality (4.3.1) is strict for $x \in \mathbb{R}\setminus[a,b]$. See e.g. [5, 10, 19, 29, 60] for the

[1]This chapter corresponds to the following paper [45]: A.B.J. Kuijlaars and P.M.J. Tibboel, The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights, J. Comput. Math. Appl. 233 (2009), 775–785.

definition of the equilibrium measure and for more information on the one-cut regular case. Under these assumptions Deift et al. [23] proved that $a_{n,n}$ and $b_{n,n}$ have asymptotic expansions in powers of $1/n$. Their approach is based on the Deift-Zhou method of steepest descent applied to the Riemann-Hilbert problem for orthogonal polynomials of Fokas, Its, and Kitaev [31]. This method was first introduced in [25] and further developed in [22, 23, 24] and many papers since then.

The asymptotic result on the recurrence coefficients was considerably refined by Bleher and Its [10, Theorem 5.2] who showed for polynomial V that there exists $\varepsilon > 0$ and real analytic functions $f_{2k}(s)$, $g_{2k}(s)$, $k = 0, 1, \ldots$, on $[1-\varepsilon, 1+\varepsilon]$ such that the asymptotic expansions

$$a_{n,N} \sim f_0\left(\frac{n}{N}\right) + \sum_{m=1}^{\infty} N^{-2m} f_{2m}\left(\frac{n}{N}\right) \tag{4.1.2}$$

$$b_{n,N} \sim g_0\left(\frac{n+1/2}{N}\right) + \sum_{m=1}^{\infty} N^{-2m} g_{2m}\left(\frac{n+1/2}{N}\right) \tag{4.1.3}$$

hold uniformly as $n, N \to \infty$ with $1-\varepsilon \leq n/N \leq 1+\varepsilon$. These $1/N^2$ expansions are used in [10] to prove the $1/N^2$ expansion of the free energy (a.k.a. logarithm of the partition function or Hankel determinant) of the associated random matrix ensemble in the one-cut regular case, see also [29].

The proof of (4.1.2) and (4.1.3) in [10] is based on the Deift et al. result referred to above, in combination with so-called string equations. It is of some interest to find a proof that is based on the Riemann-Hilbert steepest descent analysis only. Here we do this for the diagonal case $n = N$, and we obtain the following.

Theorem 4.1.1. Let V be real analytic and one-cut regular. Then there exist constants α_{2m} and β_m, $m = 1, 2, \ldots$ (depending on V) such that $a_{n,n}$ and $b_{n,n}$ have the following asymptotic expansions as $n \to \infty$:

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \qquad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m}. \tag{4.1.4}$$

The first coefficient β_1 in the expansion for $b_{n,n}$ is given explicitly by

$$\beta_1 = \frac{1}{2\pi(b-a)}\left(\frac{1}{h(b)} - \frac{1}{h(a)}\right) \tag{4.1.5}$$

where h is the function appearing in the expression (4.1.1) for the equilibrium measure associated with V.

In our proof of Theorem 4.1.1 we follow the main lines of the steepest descent analysis of [23]. We will deduce that the odd powers in the expansion of $a_{n,n}$ vanish from the structure of the local Airy parametrices around the endpoints. The expression (4.1.5) for β_1 is new, although it is likely that it can be deduced from the approach of [10] as well. The explicit formula (4.1.5) shows that $\beta_1 = 0$ if and only if $h(a) = h(b)$. It is very easy to construct examples of one-cut regular V such that $h(a) \neq h(b)$ and so $\beta_1 \neq 0$. We have thus corrected an error in a paper of Albeverio, Pastur, and Shcherbina [5, Theorem 1, formula (1.34)] who claim that $\beta_1 = 0$ always in the one-cut regular case.

Example 4.1.2. We may explicitly check Theorem 4.1.1 using Jacobi polynomials with varying parameters $\alpha = AN$, $\beta = BN$, $A, B > 0$. These polynomials are orthogonal with weight $(1-x)^{AN}(1+x)^{BN}$ on $[-1,1]$. The equilibrium measure takes the form (4.1.1) with

$$a, b = \frac{B^2 - A^2 \pm 4\sqrt{(1+A+B)(1+A)(1+B)}}{(2+A+B)^2} \tag{4.1.6}$$

and

$$h(x) = \frac{2+A+B}{2\pi(1-x^2)}, \tag{4.1.7}$$

see [59, 48]. We are in the one-cut regular case, but for weights restricted to $[-1,1]$. An analysis of the proof of Theorem 4.1.1, however, will show that the results (4.1.4)-(4.1.5) remain valid in this case as well.

From the explicit form of the recurrence coefficients for Jacobi polynomials, see e.g. [13, 48],

$$\begin{aligned} a_{n,n} &= \frac{4(1+A+B)(1+A)(1+B)}{((2+A+B)^2 - \frac{1}{n^2})(2+A+B)^2} \\ b_{n,n} &= \frac{B^2 - A^2}{(2+A+B)(2+A+B+\frac{2}{n})}, \end{aligned}$$

it is easy to see that (4.1.4) holds. Using (4.1.6)-(4.1.7) we can also ascertain the validity of (4.1.5).

4.2 The Riemann-Hilbert Problem

The Riemann-Hilbert problem for orthogonal polynomials was introduced by Fokas, Its, and Kitaev [31]. It asks for a 2×2 matrix valued function $Y(z)$ satisfying

$$\begin{cases} Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x \in \mathbb{R} \\ Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} & \text{as } z \to \infty. \end{cases} \tag{4.2.1}$$

The unique solution of (4.2.1) is (see e.g. [19])

$$Y(z) = \begin{pmatrix} \kappa_{n,N}^{-1} p_{n,N}(z) & \frac{1}{2\pi i \kappa_{n,N}} \int\limits_{\mathbb{R}} \frac{p_{n,N}(t) e^{-NV(t)}}{t-z} dt \\ -2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int\limits_{\mathbb{R}} \frac{p_{n-1,N}(t) e^{-NV(t)}}{t-z} dt \end{pmatrix}$$

where $p_{n,N}(x) = \kappa_{n,N} \pi_{n,N}(x)$ is the nth degree orthonormal polynomial. The recurrence coefficients are expressed as follows in terms of the solution of the Riemann-Hilbert problem (4.2.1), see [19, 27].

Proposition 4.2.1. Let

$$Y(z) = \left(I + \frac{1}{z} Y_1 + \frac{1}{z^2} Y_2 + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{4.2.2}$$

Then

$$a_{n,N} = (Y_1)_{12} (Y_1)_{21} \tag{4.2.3}$$

and

$$b_{n,N} = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22} \tag{4.2.4}$$

For the remainder of this chapter we will take $N = n$. We closely follow [19, 23] in applying the Deift-Zhou method of steepest descent for Riemann-Hilbert problems to (4.2.1).

4.3 The Deift-Zhou method of steepest descent

The goal of the Deift-Zhou method of steepest descent for Riemann-Hilbert problems is to change the original problem into a problem for which the asymptotics for $z \to \infty$ are normalised and for which all matrices, jump matrices and solutions alike, are asymptotically close to the identity matrix for large n which can be solved iteratively. The specific details and steps needed to achieve this goal shall be explained below.

4.3.1 The First Step: Transformation $Y \mapsto T$

The key aspect of the first step of the analysis is the equilibrium measure μ_V corresponding to V. This equilibrium measure μ_V is the unique probability measure that satisfies for some l,

$$2 \int \log |x-y|^{-1} \, d\mu_V(y) + V(x) \geq l, \qquad \text{for all } x \in \mathbb{R}, \tag{4.3.1}$$

$$2 \int \log |x-y|^{-1} \, d\mu_V(y) + V(x) = l, \qquad \text{for all } x \in \operatorname{supp} \mu_V. \tag{4.3.2}$$

For the one-cut regular case that we are considering we have that the support is one interval $[a,b]$ and $d\mu_V(x) = \psi_V(x)\,dx$ as in (4.1.1). In addition the inequality (4.3.1) is strict for $x \in \mathbb{R} \setminus [a,b]$.

Define

$$g(z) = \int \log(z-s)\,d\mu_V(s) = \int \log(z-s)\psi_V(s)\,ds \tag{4.3.3}$$

and

$$\phi(z) = \pi \int_b^z ((s-b)(s-a))^{\frac{1}{2}}\,h(s)\,ds, \quad z \in \mathbb{C} \setminus (-\infty, b] \tag{4.3.4}$$

$$\tilde{\phi}(z) = \pi \int_a^z ((s-b)(s-a)))^{\frac{1}{2}}\,h(s)\,ds, \quad z \in \mathbb{C} \setminus [a, +\infty). \tag{4.3.5}$$

If we now put

$$T(z) = e^{n(l/2)\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-n(l/2)\sigma_3}, \tag{4.3.6}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the third Pauli matrix, then T satisfies the Riemann-Hilbert problem

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ T_+(x) = T_-(x) J_T(x) \text{ for } x \in \mathbb{R}, \\ T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty, \end{cases} \tag{4.3.7}$$

where

$$J_T(x) = \begin{cases} \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x < a, \\ \begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} & \text{for } x \in (a,b), \\ \begin{pmatrix} 1 & e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x > b. \end{cases} \tag{4.3.8}$$

Since the inequality in (4.3.1) is strict for $x < a$ and $x > b$ we have that $\tilde{\phi}(x) > 0$ for $x < a$ and $\phi(x) > 0$ for $x > b$. Thus the jump matrices for T on $(-\infty, a)$ and (b, ∞) tend to the identity matrix as $n \to \infty$.

4.3.2 The Second Step: Transformation $T \mapsto S$

The second transformation is the so-called *opening of the lens* and it is based on the factorisation

$$\begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_+(x)} & 1 \end{pmatrix} \tag{4.3.9}$$

of the jump matrix J_T on (a, b). The factorisation (4.3.9) allows us to split the jump on (a, b) as shown in Figure 4.1.

We use Σ_1 and Σ_2 to denote the upper and lower lips of the lens, respectively. We define S as follows:

- For z outside the lens, we put $S = T$.
- For z within the region enclosed by Σ_1 and (a, b),
$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi} & 1 \end{pmatrix}. \tag{4.3.10}$$
- For z within the region enclosed by Σ_2 and (a, b),
$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix}. \tag{4.3.11}$$

Then S satisfies the following Riemann-Hilbert problem:

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2) \\ S_+(z) = S_-(z) J_S(z) \text{ for } z \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \\ S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{4.3.12}$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} & \text{for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (a, b), \\ \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z < a, \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z > b, \end{cases} \tag{4.3.13}$$

We may (and do) assume that the lips of the lens are in the region where $\text{Re}\,\phi < 0$, so that the jump matrices for S on Σ_1 and Σ_2 tend to the identity matrix as $n \to \infty$.

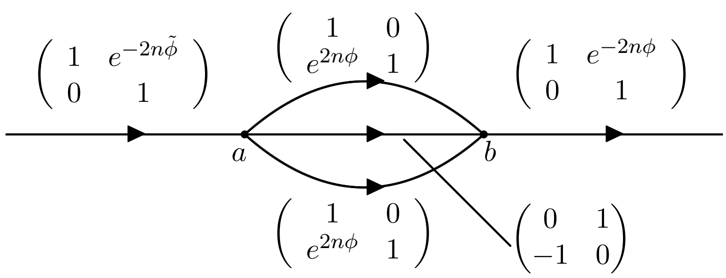

Figure 4.1: Jump matrices for S after opening of the lens

4.3.3 The Third Step: Parametrix Away From Endpoints

The parametrix away from the endpoints is a 'global solution' $N(z)$ satisfying the Riemann-Hilbert problem

$$\begin{cases} N(z) \text{ is analytic in } \mathbb{C} \setminus [a,b] \\ N_+(x) = N_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } x \in (a,b) \\ N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{4.3.14}$$

which has a solution (see [19])

$$N(z) = \begin{pmatrix} \frac{\beta(z)+\beta^{-1}(z)}{2} & \frac{\beta(z)-\beta^{-1}(z)}{2i} \\ -\frac{\beta(z)-\beta^{-1}(z)}{2i} & \frac{\beta(z)+\beta^{-1}(z)}{2} \end{pmatrix} \tag{4.3.15}$$

where $\beta(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}}$.

4.3.4 The Fourth Step: Parametrices Near Endpoints

Having constructed the 'global solution', the next step is finding 'local solutions' close to the endpoints a and b. Near b, the local situation is described as in the left picture of Figure 4.2 with jump matrix

$$J_P(z) = J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} \text{ on } \Sigma_1 \cap U \text{ and } \Sigma_2 \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ on } (a,b) \cap U \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \text{ on } (b,\infty) \cap U \end{cases}$$

where U is a (small) disk around b.

We therefore want to find a matrix function P, that solves

$$\begin{cases} P(z) \text{ is analytic on } U \setminus (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \\ P_+(z) = P_-(z) J_P(z) \text{ on } (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U \\ P(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U \end{cases} \tag{4.3.16}$$

Then $P(z)e^{n\phi(z)\sigma_3}$ should have constant jumps on $(\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U$, namely

$$\left(P(z)e^{n\phi(z)\sigma_3}\right)_+ = \left(P(z)e^{n\phi(z)\sigma_3}\right)_- \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } z \in (\Sigma_1 \cup \Sigma_2) \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } z \in (a,b) \cap U \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } z \in (b,\infty) \cap U \end{cases}$$

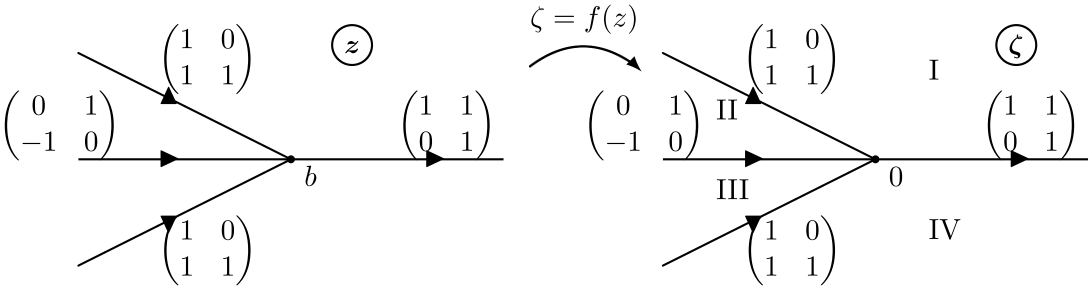

Figure 4.2: Mapping of neighbourhood of b onto a neighbourhood of $f(b) = 0$

Shrinking U if necessary, we have that

$$\zeta = f(z) = \left(\frac{3}{2}\phi(z)\right)^{2/3}$$

defines a conformal map from U to a convex neighborhood of $\zeta = 0$. We may and do assume that the lips of the lens are taken so that $\Sigma_1 \cap U$ is mapped into $\arg\zeta = 2\pi/3$, and $\Sigma_2 \cap U$ is mapped into $\arg\zeta = 2\pi/3$, see Figure 4.2. Denoting the sectors in the ζ-plane by I, II, III, IV as in Figure 4.2, and using the usual Airy function $\mathrm{Ai}(\zeta)$, we construct the Airy model solution Φ by

$$\Phi(\zeta) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \omega\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\mathrm{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector IV} \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & -\omega\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector I} \\ \begin{pmatrix} -\omega\mathrm{Ai}(\omega\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ -\omega^2\mathrm{Ai}'(\omega\zeta) & -\omega\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector II} \\ \begin{pmatrix} -\omega^2\mathrm{Ai}(\omega^2\zeta) & \omega\mathrm{Ai}(\omega\zeta) \\ -\omega\mathrm{Ai}'(\omega^2\zeta) & \omega^2\mathrm{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector III} \end{cases}$$

where $\omega = e^{2\pi i/3}$. Then Φ has the jump matrices in the ζ-plane indicated in the right-hand side of Figure 4.2.

Then for any analytic prefactor $E_n(z)$ we have that

$$P(z) = E_n(z)\Phi(n^{\frac{2}{3}}f(z))e^{n\phi(z)\sigma_3} \tag{4.3.17}$$

has the required jump matrices J_P. If we choose

$$E_n(z) = \sqrt{\pi}N(z)\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\left(n^{2/3}f(z)\right)^{\sigma_3/4} \tag{4.3.18}$$

then the matching condition $P(z) = N(z)(I+\mathcal{O}(1/n))$ as $n \to \infty$ for $z \in \partial U$, is satisfied as well, see e.g. [11, 19, 23] for further detail.

A similar construction yields a parametrix $\tilde{P}$ in a small disc $\tilde{U}$ around a. One can see that $\tilde{P}$ can be obtained by taking P and interchanging a and b and conjugating with σ_3.

4.3.5 The Fifth Step: Transformation $S \mapsto R$

Using the parametrices N, P, and $\tilde{P}$, we define the third transformation $S \mapsto R$ as follows

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ S(z)P(z)^{-1} & \text{for } z \in U \\ S(z)\tilde{P}(z)^{-1} & \text{for } z \in \tilde{U} \end{cases} \tag{4.3.19}$$

Then R has no jump on $[a, b] \setminus \overline{(U \cup \tilde{U})}$, as the jumps of S and N^{-1} cancel out. In U and $\tilde{U}$ the jumps of S cancel out with the jumps of P and $\tilde{P}$, leaving only jumps for R on the contour Σ_R shown in Figure 4.3.

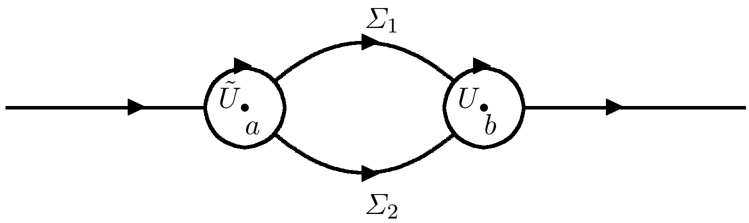

Figure 4.3: Contour Σ_R for the Riemann-Hilbert problem for R

The Riemann-Hilbert problem for R is

$$\begin{cases} R(z) \text{ is analytic on } \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z)J_R(z) \text{ for } z \in \Sigma_R \\ R(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases}$$

where

$$J_R(z) = \begin{cases} N(z)J_S(z)N(z)^{-1} \text{ for } z \in \Sigma_R \setminus (\partial U \cup \partial \tilde{U}) \\ P(z)N(z)^{-1} \text{ for } z \in \partial U \\ \tilde{P}(z)N(z)^{-1} \text{ for } z \in \partial \tilde{U} \end{cases}$$

The jump matrices $J_R(z) = N(z)J_S(z)N(z)^{-1}$ tend to the identity matrix at an exponential rate as $n \to \infty$. The jump matrices on ∂U and $\partial \tilde{U}$ tend to the identity matrix but at a slower rate of $1/n$ as $n \to \infty$. The precise form is obtained from the asymptotic expansion of the Airy function as $z \to \infty$, $-\pi < \arg z < \pi$, (see [37])

$$\operatorname{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)! \Gamma\left(\frac{1}{2}\right)} \frac{1}{z^{\frac{3}{2}k}} \tag{4.3.20}$$

and the corresponding asymptotic expansion for $\operatorname{Ai}'(z)$. Using these facts in the parametrix P we find an asymptotic expansion for the jump of R on

∂U

$$J_R(z) = P(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k}\Delta_k(z) \tag{4.3.21}$$

where

$$\begin{aligned}\Delta_k(z) =& \frac{1}{\sqrt{\pi}}\left(\frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!} - \frac{\Gamma\left(3k-\frac{3}{2}\right)}{4\cdot 9^{k-1}(2(k-1))!}\right)\frac{1}{\left(\frac{3}{2}\phi(z)\right)^k}I \\ &- \frac{1}{4\sqrt{\pi}}\frac{\Gamma\left(3k-\frac{3}{2}\right)}{9^{k-1}(2(k-1))!}\frac{1}{\left(\frac{3}{2}\phi(z)\right)^k}\sigma_2 \quad \text{for } k \text{ even}\end{aligned} \tag{4.3.22}$$

and

$$\begin{aligned}\Delta_k(z) =& -\frac{\beta(z)^2}{\left(\frac{3}{2}\phi(z)\right)^k}\frac{1}{2\sqrt{\pi}}\left(\frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!} - \frac{\Gamma\left(3k-\frac{3}{2}\right)}{2\cdot 9^{k-1}(2(k-1))!}\right)(\sigma_3 + i\sigma_1) \\ &- \frac{\beta(z)^{-2}}{\left(\frac{3}{2}\phi(z)\right)^k}\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!}(\sigma_3 - i\sigma_1) \quad \text{for } k \text{ odd}\end{aligned} \tag{4.3.23}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.3.24}$$

are the Pauli matrices.

A similar expansion

$$J_R(z) = \tilde{P}(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k}\tilde{\Delta}_k(z) \tag{4.3.25}$$

holds for the jump matrix on $\partial\tilde{U}$.

As a result we find by the methods of [23], see also [43, Lemma 8.3],

Lemma 4.3.1. There exist matrix valued functions $R_k(z)$ with the property that for every $l \in \mathbb{N}$, there exist constants $C > 0$ and $r > 0$ such that for every z with $|z| \geq r$,

$$\left\| R(z) - I - \sum_{k=1}^{l} \frac{R_k(z)}{n^k} \right\| \leq \frac{C}{|z|n^{l+1}} \tag{4.3.26}$$

So we write

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} R_k(z) \tag{4.3.27}$$

From (4.3.27), (4.3.21) and (4.3.25) and the Riemann-Hilbert problem for R, we find an additive Riemann-Hilbert problem for $R_k(z)$,

$$\begin{cases} R_k(z) \text{ is analytic on } \mathbb{C} \setminus (\partial U \cup \partial \tilde{U}) \\ R_{k+}(z) = R_{k-}(z) + \sum\limits_{l=0}^{k-1} R_{l-}(z)\Delta_{k-l}(z) \text{ for } z \in \partial U \\ R_{k+}(z) = R_{k-}(z) + \sum\limits_{l=0}^{k-1} R_{l-}(z)\tilde{\Delta}_{k-l}(z) \text{ for } z \in \partial \tilde{U} \\ R_k(z) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty \end{cases} \tag{4.3.28}$$

where $R_0(z) = I$. These Riemann-Hilbert problems can be successively solved using the Sokhotskii-Plemelj formula, or using a technique based on Laurent series expansion as in [43].

4.4 Proof of Theorem 4.1.1

For the proof of (4.1.4) we do not need to compute the explicit forms of the R_k's. However, we need to know that they have the following structure. Recall that the Pauli matrices are given in (4.3.24).

Lemma 4.4.1. For k odd, $R_k(z)$ is a linear combination of σ_1 and σ_3 and for k even, $R_k(z)$ is a linear combination of I and σ_2.

Proof. For $k = 1$, we know because of (4.3.28) that $R_{1+} = R_{1-} + \Delta_1$ on ∂U and $R_{1+} = R_{1-} + \tilde{\Delta}_1$ on $\tilde{\partial} U$. As Δ_1, $\tilde{\Delta}_1 \in \operatorname{span}\{\sigma_1, \sigma_3\}$ on account of (4.3.23), $R_1(z)$ must be a linear combination of σ_1 and σ_3 as well.

Let $k \geq 1$ and once more observe (4.3.28). If k is odd, then again by (4.3.23) Δ_k, $\tilde{\Delta}_k \in \operatorname{span}\{\sigma_1, \sigma_3\}$ and using induction on k, for every $l < k$, $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are products of a linear combination of σ_1 and σ_3 and a linear combination of I and σ_2 (see also (4.3.22)–(4.3.23)), which results in a linear combination of σ_1 and σ_3. Thus all terms in the (additive) jump for R_k on ∂U and on $\partial \tilde{U}$ are in the span of σ_1 and σ_3, and it follows that $R_k \in \operatorname{span}\{\sigma_1, \sigma_3\}$ if k is odd.

If k is even, then by induction, where we use again ((4.3.22)–(4.3.23), we have that $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are either products of two linear combinations of I and σ_2 (in case l is even), or products of two linear combinations of σ_1 and σ_3 (in case l is odd). In both cases we find that $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are linear combinations of I and σ_2, which implies that $R_k \in \operatorname{span}\{I, \sigma_2\}$ if k is even. $\square$

Now we can finally prove our main result.

Proof of Theorem 4.1.1. We start from the expressions (4.2.3) and (4.2.4) for $a_{n,n}$ and $b_{n,n}$ in terms of the solution of the Riemann-Hilbert problem for Y. Following the transformations $Y \mapsto T \mapsto S$, we find that

$$a_{n,n} = (S_1)_{12}\,(S_1)_{21} \tag{4.4.1}$$

and

$$b_{n,n} = \frac{(S_2)_{12}}{(S_1)_{12}} - (S_1)_{22} \tag{4.4.2}$$

where S_1 and S_2 are the terms in the expansion of $S(z)$ as $z \to \infty$,

$$S(z) = I + \frac{1}{z}S_1 + \frac{1}{z^2}S_2 + \mathcal{O}\left(\frac{1}{z^3}\right).$$

To obtain (4.4.2) we use that $g(z) = \log z + \mathcal{O}(1/z)$, see also [27].

By (4.3.19), we know that $S(z) = R(z)N(z)$ for $|z|$ large enough, so we need the first terms in the expansions of $N(z)$ and $R(z)$ as $z \to \infty$. From (4.3.15) we have

$$\begin{aligned} N(z) &= \frac{\beta(z)+\beta(z)^{-1}}{2}I + \frac{\beta(z)-\beta(z)^{-1}}{2}\sigma_2 \\ &= I - \frac{(b-a)}{4}\sigma_2\frac{1}{z} + \left(\frac{(b-a)^2}{32}I - \frac{b^2-a^2}{8}\sigma_2\right)\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{4.4.3}$$

and from Lemma 4.4.1

$$\begin{aligned} R(z) = I &+ \frac{1}{z}\left(\sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m1\sigma_1}\sigma_1 + R_{m1\sigma_3}\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m1I}I + R_{m1\sigma_2}\sigma_2\right)\right) \\ &+ \frac{1}{z^2}\left(\sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m2\sigma_1}\sigma_1 + R_{m2\sigma_3}\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m2I}I + R_{m2\sigma_2}\sigma_2\right)\right) \\ &+ \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{4.4.4}$$

where the constants R_{mjI}, $R_{mj\sigma_k}$, for $m \in \mathbb{N}$, $j = 1, 2$, and $k = 1, 2, 3$ are such that $R_{mjI}I + \sum_{k=1}^{3} R_{mj\sigma_k}\sigma_k$ is the coefficient of z^{-j} in the Laurent expansion of $R_m(z)$ around $z = \infty$.

Therefore, by (4.4.3) and (4.4.4),

$$\begin{aligned} S(z) =&R(z)N(z) \sim I + \frac{1}{z}\Bigg(-\frac{(b-a)}{4}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m1\sigma_1}\sigma_1 + R_{m1\sigma_3}\sigma_3\right) \\ &+ \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m1I}I + R_{m1\sigma_2}\sigma_2\right)\Bigg) \\ &+ \frac{1}{z^2}\Bigg(\frac{(b-a)^2}{32}I - \frac{b^2-a^2}{8}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m}\Bigg(\left(R_{m2\sigma_1} + i\frac{b-a}{4}R_{m1\sigma_3}\right)\sigma_1 \\ &+ \left(R_{m2\sigma_3} - i\frac{b-a}{4}R_{m1\sigma_1}\right)\sigma_3\Bigg) + \sum_{m \text{ even}} \frac{1}{n^m}\Bigg(\left(R_{m2I} - \frac{b-a}{4}R_{m1\sigma_2}\right)I \\ &+ \left(R_{m2\sigma_2} - \frac{b-a}{4}R_{m1I}\right)\sigma_2\Bigg)\Bigg) + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{4.4.5}$$

which implies that

$$(S_1)_{12} \sim \frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m}R_{m1\sigma_1} - i\sum_{m \text{ even}} \frac{1}{n^m}R_{m1\sigma_2} \tag{4.4.6}$$

and

$$(S_1)_{21} \sim -\frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m}R_{m1\sigma_1} + i\sum_{m \text{ even}} \frac{1}{n^m}R_{m1\sigma_2} \tag{4.4.7}$$

Inserting (4.4.6) and (4.4.7) into (4.4.1) then finally gives

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}$$

for certain constants α_{2m}.

Similar to (4.4.6) and (4.4.7) we have that $(S_2)_{12}$ and $(S_1)_{22}$ have asymptotic expansions in powers of $1/n$. From the expansion (4.4.5) for S, we see

$$\begin{aligned} (S_2)_{12} \sim &\frac{b^2-a^2}{8}i + \sum_{m \text{ odd}} \frac{1}{n^m}\left(\frac{b-a}{4}iR_{m1\sigma_3} + R_{m2\sigma_1}\right) \\ &+ \sum_{m \text{ even}} \frac{1}{n^m}i\left(\frac{b-a}{4}R_{m1I} - R_{m2\sigma_2}\right) \end{aligned}$$

and

$$(S_1)_{22} \sim -\sum_{m \text{ odd}} \frac{1}{n^m}R_{m1\sigma_3} + \sum_{m \text{ even}} \frac{1}{n^m}R_{m1I}$$

From (4.4.2) it then follows that

$$b_{n,n} \sim \sum_{m=0}^{\infty} \frac{\beta_m}{n^m} \tag{4.4.8}$$

where $\beta_0 = \frac{b+a}{2}$ and

$$\beta_1 = 2R_{11\sigma_3} - \frac{4}{b-a} iR_{12\sigma_1} + \frac{2(b+a)}{b-a} iR_{11\sigma_1}. \tag{4.4.9}$$

Our final task is to further evaluate the right-hand side of (4.4.9). As in [43], we have that Δ_1 is meromorphic in a neighborhood of b with a pole in b. Indeed, if we write

$$\frac{\beta(z)^{-2}}{\phi(z)} = (z-b)^{-2} \sum_{m=0}^{\infty} B_m (z-b)^m, \qquad B_0 = \frac{3}{2\pi h(b)}, \tag{4.4.10}$$

and use (4.3.23), then we find for z in a neighborhood of b,

$$\begin{aligned} \Delta_1(z) = &\left(-\frac{5B_1}{144} (\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)} (\sigma_3 + i\sigma_1) \right) \frac{1}{z-b} \\ &- \frac{5B_0}{144} (\sigma_3 - i\sigma_1) \frac{1}{(z-b)^2} + \mathcal{O}(1). \end{aligned} \tag{4.4.11}$$

Similarly, for z in a neighborhood of a, we have

$$\frac{\beta(z)^{2}}{\tilde{\phi}(z)} = (z-a)^{-2} \sum_{m=0}^{\infty} A_m (z-a)^m, \qquad A_0 = \frac{3}{2\pi h(a)}, \tag{4.4.12}$$

and

$$\begin{aligned} \tilde{\Delta}_1(z) = &\left(-\frac{5A_1}{144} (\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)} (\sigma_3 - i\sigma_1) \right) \frac{1}{z-a} \\ &- \frac{5A_0}{144} (\sigma_3 + i\sigma_1) \frac{1}{(z-a)^2} + \mathcal{O}(1). \end{aligned} \tag{4.4.13}$$

As in [43] we have that $R_1(z)$ for $z \in \mathbb{C} \setminus \overline{U \cup \tilde{U}}$ is equal to the sum of the Laurent parts of (4.4.11) and (4.4.13). Expanding $R_1(z)$ as $z \to \infty$, we then get

$$R_1(z) = R_{11} \frac{1}{z} + R_{12} \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \quad \text{as } z \to \infty,$$

where

$$
\begin{aligned}
R_{11} = &- \frac{5A_1}{144}\,(\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)}\,(\sigma_3 - i\sigma_1) \\
&- \frac{5B_1}{144}\,(\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)}\,(\sigma_3 + i\sigma_1) \\
R_{12} = &- \frac{5aA_1}{144}\,(\sigma_3 + i\sigma_1) - \frac{7aA_0}{144(b-a)}\,(\sigma_3 - i\sigma_1) \\
&- \frac{5bB_1}{144}\,(\sigma_3 - i\sigma_1) + \frac{7bB_0}{144(b-a)}\,(\sigma_3 + i\sigma_1) \\
&- \frac{5A_0}{144}\,(\sigma_3 + i\sigma_1) - \frac{5B_0}{144}\,(\sigma_3 - i\sigma_1)\,.
\end{aligned}
$$

Thus

$$
R_{11\sigma_3} = -\frac{5(A_1 + B_1)}{144} - \frac{7(A_0 - B_0)}{144(b-a)}, \tag{4.4.14}
$$

$$
R_{11\sigma_1} = -i\frac{5(A_1 - B_1)}{144} + i\frac{7(A_0 + B_0)}{144(b-a)}, \tag{4.4.15}
$$

$$
R_{12\sigma_1} = -i\frac{5(aA_1 - bB_1)}{144} + i\frac{7(aA_0 + bB_0)}{144(b-a)} - i\frac{5(A_0 - B_0)}{144}. \tag{4.4.16}
$$

Inserting (4.4.14)–(4.4.16) into (4.4.9), we find after straightforward calculations that A_1 and B_1 fully disappear and that (4.4.9) reduces to

$$
\beta_1 = \frac{B_0 - A_0}{3(b-a)}.
$$

Using the explicit formulas for A_0 and B_0 given in (4.4.10) and (4.4.12), we arrive at (4.1.5), which completes the proof of Theorem 4.1.1. □

Chapter 5

Limit behaviour of reproducing kernels with respect to a non-analytic weight

5.1 Introduction

Previous chapters suggest that whenever dealing with orthogonal polynomials with respect to an analytic weight, Deift-Zhou steepest descent analysis allows us to deduce asymptotic behaviour of both polynomials and related mathematical objects. What, however, can be said about non-analytic weights?
In this chapter, strongly inspired by Lubinsky's papers [54] and [53], our aim is twofold:

- To generalise a variety of results on the limit behaviour of reproducing kernels with respect to analytic weights to the case that the weights are continuous.

- To introduce a streamlined way of attacking these problems.

This way, it becomes possible to have the Deift-Zhou method of steepest descent deal with the analytic case and then use our generalisation argument, resulting in a strategy that should be able to deal with continuous weights instead of analytic ones.

Throughout this chapter, we adopt the following notation:

- Let μ be a finite, positive Borel measure on a subset $\Omega \subset \mathbb{R}$. We define *orthonormal polynomials* $\{p_n\}_{n=0}^{\infty}$ through the orthogonality condition

$$\int_{\Omega} p_i(x)p_j(x)d\mu(x) = \delta_{ij} \tag{5.1.1}$$

where

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

In this chapter, Ω will be characteristically $[-1, 1]$ or $\mathbb{R}$.

- We define a weight function w through $d\mu(x) = w(x)dx$.
- We define reproducing kernels $\mathcal{K}_n(x, y)$ through

$$\mathcal{K}_n(x, y) = \sum_{k=0}^{n-1} p_k(x)p_k(y) \tag{5.1.2}$$

- We define normalised reproducing kernels $K_n(x, y)$ through

$$K_n(x, y) = \mathcal{K}_n(x, y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}} \tag{5.1.3}$$

5.2 Main results

We will split our results into two sections

- The first section will relate to limit behaviour for fixed weights.
- The second section will relate to limit behaviour for varying weights

5.2.1 Limit behaviour for reproducing kernels with respect to a fixed weight.

Lubinsky showed in [54] that

Theorem 5.2.1. Let μ be a finite positive Borel measure on $(-1, 1)$ that is regular (see [63] for a definition) with weight w. Let I be a closed subinterval of $(-1, 1)$ such that μ is absolutely continuous in an open interval containing I. Assume that the weight w is positive and continuous in I. Then uniformly for $x \in I$ and a, b in compact subsets of the real line, we have

$$\lim_{n\to\infty} \frac{\mathcal{K}_n\left(x + \frac{a}{K_n(x,x)}, x + \frac{b}{K_n(x,x)}\right)}{\mathcal{K}_n(x, x)} = \frac{\sin \pi(a - b)}{\pi(a - b)}$$

Theorem 5.2.1 signified a breakthrough in the sense that up to its appearance only results for analytic, or piecewise analytic weights or even differentiable weights were known (see for example [9], [19], [20], [46], [55], [58] and many more).

The paper [54] was preceded by [53], in which it was shown that the same method applied to edge behaviour as well. In this case, working with the *Jacobi weight*

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta \tag{5.2.1}$$

(see for example [43], [46]), where $\alpha, \beta > -1$ and $x \in (-1,1)$, the limit behaviour was related to the so-called *Bessel kernel* (see [46])

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(x^{1/2})y^{1/2}J'_\alpha(y^{1/2}) - x^{1/2}J'_\alpha(x^{1/2})J_\alpha(y^{1/2})}{2(x-y)}$$

where J_α is the Bessel function of order α. Specifically, the result of [53] is

Theorem 5.2.2. Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$ and in J, its absolutely continuous component has the form $w = hw^{(\alpha,\beta)}$. Assume that $h(1) > 0$ and h is continuous at 1. Then uniformly for a, b in compact subsets of $(0,\infty)$, we have

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n\left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a,b)$$

Our technique essentially takes Lubinsky's method and departs from results for analytic weights instead, reducing the entire process to a simple ϵ-argument. To illustrate, we will first prove that

Theorem 5.2.3. Let μ define a positive Borel measure on $(-1,1)$ through $d\mu(x) = h(x)w^{\alpha,\beta}(x)dx$, where $h(x)$ is continuous and positive on $[-1,1]$. Let $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$. For x in a closed subinterval of $(-1,1)$ and u, v in compact subsets of $\mathbb{R}$, we have that uniformly

$$\lim_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}. \tag{5.2.2}$$

Secondly, to show that edge behaviour can be dealt with just as easily, we will prove that

Theorem 5.2.4. Let μ define a positive Borel measure on $(-1,1)$ through $d\mu(x) = h(x)w^{(\alpha,\beta)}(x)dx$, $\alpha, \beta > -1$, where $h(x) > 0$ and $h(x)$ is continuous for $x \in [-1,1]$ Then for u, v in compact subsets of $(0,\infty)$, we have that uniformly

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n\left(1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}\right) = \mathbb{J}_\alpha(u,v)$$

Thirdly, we prove that for a continuous weight with a jump the limit behaviour is dependent on the confluent hypergeometric kernel

$$\mathbb{K}_c^{CHF}(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda;2\pi ix);G(\lambda;2\pi iy)] \qquad (5.2.3)$$

where $\lambda = \frac{i\log c}{\pi}$, $G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$, with

$$[f(x);g(y)] = f(x)g(y) - f(y)g(x)$$

and

$$\nu_0(x) = \begin{cases} 1 \text{ if } x < 0 \\ c^2 > 0 \text{ if } x \geq 0 \end{cases}$$

and where $\phi(a,c;z)$ as in (A.6.1) a solution to the confluent hypergeometric equation

$$zy''(z)) + (c-z)y'(z) - ay(z) = 0$$

This kernel was introduced, in a slightly different fashion, by Foulquié Moreno, Martínez-Finkelshtein and Soussa in [32], regarding the limit behaviour of $\mathcal{K}_n$ (see (5.1.2)) for the case that

$$w(x) = h(x)w^{\alpha,\beta}(x)\nu_0(x) \text{ with } x \in (-1,1).$$

We will be using (5.2.3) instead, as it is more convenient when discussing the limit behaviour for a normalised reproducing kernel with respect to a weight with a jump in x_0. Based on the analysis in [32], it should be noted that (5.2.3) is simply a different way of writing down the same result. See chapter 2, Theorem 2.1.1 for details.
From hereon, let

$$\nu_{x_0}(x) = \begin{cases} 1 \text{ if } x < x_0 \\ c^2 > 0 \text{ if } x \geq x_0 \end{cases} \qquad (5.2.4)$$

where $x_0 \in (-1,1)$.
Our theorem is:

Theorem 5.2.5. Let μ define a positive Borel measure on $(-1,1)$ through $d\mu(x) = w(x)dx = h(x)\nu_{x_0}(x)w^{(\alpha,\beta)}(x)dx$, $\alpha,\beta > -1$, where h is a positive and continuous function. Then for x in a closed subinterval of $(-1,1)$ and u, v in compact subsets of $\mathbb{R}$ we have that uniformly

$$\lim_{n\to\infty}\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,v)$$

where $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$

Of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, the last one is a new result, whereas Theorem 5.2.3 and Theorem 5.2.4 are weaker versions of Theorem 5.2.1 and Theorem 5.2.2.
Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, can be further refined into

Theorem 5.2.6. Let c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive finite Borel measure on $(-1,1)$ through $d\mu(x) = h(x)w^{\alpha,\beta}(x)dx$, where

$$c_1 \leq h(x) \leq c_2$$

for $x \in [-1,1]$ and continuous on an open subinterval $I \subset (-1,1)$.
Let $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$. For x in a closed subinterval of $(-1,1)$ and u, v in compact subsets of $\mathbb{R}$, we have that uniformly

$$\lim_{n\to\infty} \frac{1}{n\xi(x)} K_n \left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)} \right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}.$$

Theorem 5.2.7. Let $\delta > 0$ and c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive Borel measure on $(-1,1)$ through $d\mu(x) = h(x)w^{(\alpha,\beta)}(x)dx$, $\alpha, \beta > -1$, where

$$c_1 \leq h(x) \leq c_2$$

for $x \in [-1,1]$ and $h(x)$ is continuous for $x \in [1-2\delta, 1] \subset [-1,1]$. Then for u, v in compact subsets of $(0,\infty)$ we have that uniformly

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n \left(1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2} \right) = \mathbb{J}_\alpha(u,v)$$

Theorem 5.2.8. Let $I \subset [-1,1]$ be an open interval, $x_0 \in I$ and let c_1, $c_2 \in \mathbb{R}_{>0}$. Let μ define a positive Borel measure on $(-1,1)$ through

$$d\mu(x) = w(x)dx = h(x)\nu_{x_0}(x)w^{(\alpha,\beta)}(x)dx$$

$\alpha, \beta > -1$, $x_0 \in (-1,1)$, where h is continuous on I and

$$c_1 \leq h(x) \leq c_2$$

for $x \in [-1,1]$. ν_{x_0} is as in (5.2.4) and $w^{(\alpha,\beta)}$ as in (5.2.1). Then for x in a closed subinterval of $(-1,1)$ and u, v in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) = \mathbb{K}_c^{CHF}(u,v)$$

where $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$

Theorem 5.2.8 is a new result, the other two are again weaker versions of Theorem 5.2.1 and Theorem 5.2.2.

Further results were obtained by Levin and Lubinsky (see [49], [50], [52]). Results for more general examples of the support Ω of μ were obtained by Simon (see [62]) and Totik (see [65]) but we will not get into them here.

5.2.2 Limit behaviour for reproducing kernels with respect to a varying weight

Limit behaviour for reproducing kernels with respect to fixed weights well covered, an obvious next step is to check how our method can be applied to deduce limit behaviour for reproducing kernels with respect to varying weights.
A vast library of results exists for weights of the form $w_N(x) = e^{-NV(x)}$, where N is a parameter approaching infinity, V is real analytic and

$$\frac{V(x)}{\log(1+x^2)} \to +\infty \tag{5.2.5}$$

if $|x|$ approaches infinity. (See for example [9], [15], [19], [20], [27] and many more, or, for slight variations on $w_N(x) = e^{-NV(x)}$, [14], [16], [17], [18], [41]).
We will show how our method can be used to generalise results about limit behaviour for weights

$$w_N(x) = e^{-NV(x)}$$

to the case that

$$w_N(x) = H(x)e^{-NV(x)}$$

where $H(x)$ is a positive valued continuous function.
Throughout this section, we will use the following notation:

- Let μ_N be a finite, positive Borel measure on $\mathbb{R}$, defined through $d\mu_N(x) = w_N(x)dx$. We define *orthonormal polynomials* $\{p_{n,N}\}_{n=0}^{\infty}$ through the orthogonality condition

$$\int_{\mathbb{R}} p_{i,N}(x)p_{j,N}(x)d\mu_N(x) = \delta_{ij} \tag{5.2.6}$$

  where

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

- We define reproducing kernels $\mathcal{K}_{n,N}(x,y)$ through

$$\mathcal{K}_{n,N}(x,y) = \sum_{k=0}^{n-1} p_{k,N}(x)p_{k,N}(y) \tag{5.2.7}$$

- We define normalised reproducing kernels $K_{n,N}(x,y)$ through

$$K_{n,N}(x,y) = \mathcal{K}_{n,N}(x,y)w_N(x)^{\frac{1}{2}}w_N(y)^{\frac{1}{2}} \tag{5.2.8}$$

Before continuing, we need to quickly review some concepts related to V. For V real analytic and fulfilling (5.2.5), a function ψ_V called the *mean eigenvalue density* exists (see [19]), that is also the density of the *equilibrium measure* μ_V, the unique Borel probability measure that solves

$$(i) \qquad 2\int \log|x-y|^{-1}d\mu_V(y) + V(x) \geq l \text{ for all } x \in \mathbb{R}$$

$$(ii) \qquad 2\int \log|x-y|^{-1}d\mu_V(y) + V(x) = l \text{ on } \{x : \psi_V(x) > 0\}$$

(see Theorem 6.132 of [19]) which we have seen before in chapter 4.
Due to the real analyticity of V, the set $\{x : \psi_V(x) > 0\}$ is in fact a finite union of intervals (see [21]). Even more, it was proven in [21], [23] that

- Near right edge points x^* of $\operatorname{supp}\psi_V = \overline{\{x : \psi_V(x) > 0\}}$, for x approaching x^* from within $\operatorname{supp}\psi_V$

  $$\psi_V(x) = c(x^* - x)^{2k+\frac{1}{2}}(1 + o(1))$$

  where $c > 0$ and k is a non-negative integer.

- Around interior points x^* of $\operatorname{supp}\psi_V$ where $\psi_V(x)$ vanishes

  $$\psi_V(x^*) = c(x^* - x)^{2k}(1 + o(1))$$

  where again $c > 0$ and k is a positive integer.

In [23] it was shown that for $\psi_V(x) > 0$ and $H(x) = 1$

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{v}{n\psi_V(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}$$

and for right edge points x^* of $\operatorname{supp}\psi_V$ where ψ_V goes to zero as a square root

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(x + \frac{u}{(cn)^{2/3}}, x + \frac{v}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

for some $c > 0$, where Ai is the Airy function (see section 1.3.2).

From hereon we will assume that

$$\operatorname{supp}\psi_V = [a, b] \tag{5.2.9}$$

on which

$$\psi_V(x) = h(x)\sqrt{(x-a)(b-x)} \tag{5.2.10}$$

for constants a, $b \in \mathbb{R}$, $a < b$ and $h(x)$ a positive analytic function.
We will prove that

Theorem 5.2.9. Let $K_{n,N}(x,y)$ be the normalised reproducing kernel with respect to a weight function $w_N(x) = H(x)e^{-NV(x)}$, where H is a positive, continuous function, V is real analytic. Then

- For $\psi_V(x) > 0$ and for u, $v \in \mathbb{R}$:

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{v}{n\psi_V(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}$$

- For b a right edge point of $\operatorname{supp}\psi_V$ and $u, v \in \mathbb{R}$:

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b + \frac{u}{(cn)^{2/3}}, b + \frac{v}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

The first result in Theorem 5.2.9 is a weaker version of [51], the second result in Theorem 5.2.9 is new.

5.3 Background theory

Before proving our main theorems, we need some background material in the form of the following lemmas, which were taken from [61]. The proofs of these lemmas rely heavily on the so-called reproductive property of $\mathcal{K}_n$ (see for example [61]), which states that for a reproducing kernel $\mathcal{K}_n(x,y)$ and any polynomial P of degree less than n,

$$\int \mathcal{K}_n(x,y)P(x)d\mu(x) = P(y) \tag{5.3.1}$$

The first lemma is an inequality due to Lubinsky and can be found in [54].

Lemma 5.3.1. [Lubinsky Inequality] Let $\mathcal{K}_n(x,y)$ and $\widehat{\mathcal{K}_n}(x,y)$ be reproducing kernels corresponding to measures μ and $\widehat{\mu}$ respectively,
with $d\mu \le d\widehat{\mu}$. Then

$$(\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y))^2 \le \mathcal{K}_n(x,x)\left(\mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)\right)$$

Proof. This proof was taken from [61] but is originally due to Lubinsky (see [54]). Note that for y fixed, $\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y)$ is a polynomial of degree $n-1$ in x. So using (5.3.1) we find that

$$\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y) = \int \mathcal{K}_n(t,x)\left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)d\mu(t)$$

Thus, by Cauchy-Schwarz,

$$\begin{aligned}\left|\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y)\right| &= \left|\int \mathcal{K}_n(t,x)\left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)d\mu(t)\right| \\ &\le \left(\int \mathcal{K}_n(t,x)^2 d\mu(t)\right)^{\frac{1}{2}} \left(\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t)\right)^{\frac{1}{2}}\end{aligned} \tag{5.3.2}$$

Reusing (5.3.1) and observing that for x fixed $\mathcal{K}_n(t,x)$ is a polynomial of degree $n-1$ in t, we find that

$$\begin{aligned}\int \mathcal{K}_n(t,x)^2 d\mu(t) &= \int \mathcal{K}_n(t,x)\mathcal{K}_n(t,x) d\mu(t)\\ &= \mathcal{K}_n(x,x)\end{aligned}$$

Furthermore, note that

$$\begin{aligned}&\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t)\\ &= \int \left(\mathcal{K}_n(t,y)^2 - 2\mathcal{K}_n(t,y)\widehat{\mathcal{K}_n}(t,y) + \widehat{\mathcal{K}_n}(t,y)^2\right) d\mu(t)\\ &= \int \mathcal{K}_n(t,y)^2 d\mu(t) - 2\int \mathcal{K}_n(t,y)\widehat{\mathcal{K}_n}(t,y) d\mu(t) + \int \widehat{\mathcal{K}_n}(t,y)^2 d\mu(t)\end{aligned}$$

Applying (5.3.1) once more, realising that for y fixed $\widehat{K_n}(t,y)$ and $\mathcal{K}_n(t,y)$ are polynomials of degree $n-1$ in t, we deduce that

$$\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t) = \mathcal{K}_n(y,y) - 2\widehat{\mathcal{K}_n}(y,y) + \int \widehat{\mathcal{K}_n}(t,y)^2 d\mu(t)$$

Recalling that $d\mu \leq d\widehat{\mu}$ we conclude that

$$\begin{aligned}\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t) &= \mathcal{K}_n(y,y) - 2\widehat{\mathcal{K}_n}(y,y) + \int \widehat{\mathcal{K}_n}(t,y)^2 d\mu(t)\\ &\leq \mathcal{K}_n(y,y) - 2\widehat{\mathcal{K}_n}(y,y) + \int \widehat{\mathcal{K}_n}(t,y)^2 d\widehat{\mu}(t)\end{aligned}$$

By the reproducing property of $\widehat{\mathcal{K}_n}$ (see (5.3.1)), we find that

$$\begin{aligned}\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t) &\leq \mathcal{K}_n(y,y) - 2\widehat{\mathcal{K}_n}(y,y) + \widehat{\mathcal{K}_n}(y,y)\\ &= \mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)\end{aligned}$$

Inserting

$$\int \mathcal{K}_n(t,x)^2 d\mu(t) = \mathcal{K}_n(x,x)$$

and

$$\int \left(\mathcal{K}_n(t,y) - \widehat{\mathcal{K}_n}(t,y)\right)^2 d\mu(t) \leq \mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)$$

into (5.3.2) we see that

$$\left(\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y)\right)^2 \leq \mathcal{K}_n(x,x)\left(\mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)\right)$$

thereby completing the proof. □

Lemma 5.3.2. Let $\mathcal{K}_n(x,y)$ and $\widehat{\mathcal{K}_n}(x,y)$ be reproducing kernels (see (5.1.2)) corresponding to weights w and $\widehat{w}$ on Ω respectively and let $K_n(x,y)$ and $\widehat{K_n}(x,y)$ be their respective normalised counterparts (see (5.1.2) and (5.1.3)). Then, if $w \leq \widehat{w}$ on Ω,

$$\begin{aligned} |K_n(x,y) - \widehat{K_n}(x,y)| \leq K_n(x,x)^{\frac{1}{2}} \left(K_n(y,y) - \widehat{K_n}(y,y) \frac{w(y)}{\widehat{w}(y)} \right)^{\frac{1}{2}} \\ + \left| \widehat{K_n}(x,y) \right| \left| \frac{w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}}{\widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}}} - 1 \right| \end{aligned} \tag{5.3.3}$$

where $x, y \in \Omega$.

Proof. Using the triangle inequality, we see that

$$\begin{aligned} |K_n(x,y) - \widehat{K_n}(x,y)| &= \left| \mathcal{K}_n(x,y) w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} - \widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} \right| \\ &\leq \left| \mathcal{K}_n(x,y) w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} - \widehat{\mathcal{K}_n}(x,y) w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} \right| \\ &\quad + \left| \widehat{\mathcal{K}_n}(x,y) w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} - \widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} \right| \end{aligned}$$

Isolating $w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}$ in the first term on the right hand side and $\left| \widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} \right|$ in the second term on the right hand side then leads to

$$\begin{aligned} |K_n(x,y) - \widehat{K_n}(x,y)| \leq \left| \mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y) \right| w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} \\ + \left| \widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} \right| \left| \frac{w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}}{\widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}}} - 1 \right| \end{aligned}$$

The Lubinsky inequality (see Lemma 5.3.1) tells us that

$$(\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y))^2 \leq \mathcal{K}_n(x,x) \left(\mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y) \right)$$

So using Lubinsky's inequality on the first term on the right hand side gives

$$\begin{aligned} |K_n(x,y) - \widehat{K_n}(x,y)| \leq \mathcal{K}_n(x,x)^{\frac{1}{2}} \left(\mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y) \right)^{\frac{1}{2}} w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} \\ + \left| \widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} \right| \left| \frac{w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}}}{\widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}}} - 1 \right| \end{aligned}$$

Using that $\mathcal{K}_n(x,x)^{\frac{1}{2}} w(x)^{\frac{1}{2}} = K_n(x,x)^{\frac{1}{2}}$, $\widehat{\mathcal{K}_n}(x,x)^{\frac{1}{2}} \widehat{w}(x)^{\frac{1}{2}} = \widehat{K_n}(x,x)^{\frac{1}{2}}$ and $\widehat{\mathcal{K}_n}(x,y) \widehat{w}(x)^{\frac{1}{2}} \widehat{w}(y)^{\frac{1}{2}} = \widehat{K_n}(x,y)$ we get (5.3.3). □

Lemma 5.3.3. If $d\mu \leq d\widehat{\mu}$, then $\widehat{\mathcal{K}}_n(x,x) \leq \mathcal{K}_n(x,x)$

Proof. By Lemma 5.3.1 we have that

$$0 \leq (\mathcal{K}_n(x,y) - \widehat{\mathcal{K}_n}(x,y))^2 \leq \mathcal{K}_n(x,x)\left(\mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)\right)$$

so

$$0 \leq \mathcal{K}_n(y,y) - \widehat{\mathcal{K}_n}(y,y)$$

which finalises the proof. □

Lemma 5.3.4. Let $w^{(-)}$, w and $w^{(+)}$ be weight functions and let $K_n^{(+)}$, K_n and $K_n^{(-)}$ be their respective normalised reproducing kernels. Furthermore, let $w^{(-)} \leq w \leq w^{(+)}$. Then

$$K_n^{(+)}(x,x)\frac{w(x)}{w^{(+)}(x)} \leq K_n(x,x) \leq K_n^{(-)}(x,x)\frac{w(x)}{w^{(-)}(x)} \tag{5.3.4}$$

Proof. Let $\mathcal{K}_n^{(\pm)}$ be the reproducing kernels corresponding to $w^{(\pm)}$.
By Lemma 5.3.3,

$$\mathcal{K}_n^{(+)}(x,x) \leq \mathcal{K}_n(x,x) \leq \mathcal{K}_n^{(-)}(x,x) \tag{5.3.5}$$

Multiplying all expressions in (5.3.5) with $w(x)$ and expressing the new statement in terms of normalised reproducing kernels then gives

$$K_n^{(+)}(x,x)\frac{w(x)}{w^{(+)}(x)} \leq K_n(x,x) \leq K_n^{(-)}(x,x)\frac{w(x)}{w^{(-)}(x)}$$

which is exactly (5.3.4). □

5.4 Limit behaviour for reproducing kernels with respect to a fixed, continuous weight

In this section we will prove Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5. The strategy of these proofs can be split into two steps:

- The first step will be to obtain our result for the diagonal case, or $K_n(x,x)$. The main idea will be to find analytic weights $w^{(-)}$, $w^{(+)}$ that lie close to w and for which $w^{(-)}(x) \leq w(x) \leq w^{(+)}(x)$. Then, taking appropriate limits, Lemma 5.3.4 will in every proof lead to the result for the diagonal case.

- The second step will then be to generalise the diagonal result to the off-diagonal result, using Lemma 5.3.2.

5.4.1 Proof of Theorem 5.2.3

The diagonal case

Let $\epsilon > 0$. Due to the Weierstrass Approximation Theorem, there exist polynomials $h^{(-)}$, $h^{(+)}$ that are positive on $[-1,1]$, such that

$$h^{(-)}(x) \leq h(x) \leq h^{(+)}(x)$$

and $|h^{(\pm)}(x) - h(x)| < \epsilon$ for $x \in [-1,1]$. Denote the kernel related to the weight $w^{(\pm)}(x) = h^{(\pm)}(x)\nu_{x_0}(x)w^{(\alpha,\beta)}(x)$ by $K_n^{(\pm)}(x,y)$. Then by Lemma 5.3.4

$$K_n^{(+)}(x,x)\frac{w(x)}{w^{(+)}(x)} \leq K_n(x,x) \leq K_n^{(-)}(x,x)\frac{w(x)}{w^{(-)}(x)}$$

for $x \in [-1,1]$.
That is

$$K_n^{(+)}(x,x)\frac{h(x)}{h^{(+)}(x)} \leq K_n(x,x) \leq K_n^{(-)}(x,x)\frac{h(x)}{h^{(-)}(x)} \tag{5.4.1}$$

as the factor $w^{(\alpha,\beta)}(x)\nu_{x_0}(x)$ cancels out.
Note that (5.4.1) holds for all $x \in [-1,1]$. Let $u \in \mathbb{R}$. As $\xi(x) > 0$ for $x \in (-1,1)$, for n large enough, $x + \frac{u}{n\xi(x)} \in (-1,1)$ and

$$\frac{1}{n\xi(x)}K_n^{(+)}\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right)\frac{h\left(x+\frac{u}{n\xi(x)}\right)}{h^{(+)}\left(x+\frac{u}{n\xi(x)}\right)} \tag{5.4.2}$$

$$\leq \frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right) \tag{5.4.3}$$

$$\leq \frac{1}{n\xi(x)}K_n^{(-)}\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right)\frac{h\left(x+\frac{u}{n\xi(x)}\right)}{h^{(-)}\left(x+\frac{u}{n\xi(x)}\right)} \tag{5.4.4}$$

As, by [46], Theorem 5.2.3 holds for analytic h, we find that particularly for $h^{(\pm)}$

$$\lim_{n\to\infty}\frac{1}{n\xi(x)}K_n^{(\pm)}\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right) = \mathbb{S}(u,u) \tag{5.4.5}$$

where $\mathbb{S}(u,v)$ is the sine kernel, which is equal to 1 for $u = v$.
Thus, taking the limit infimum for $n \to \infty$ in (5.4.2) and (5.4.3) gives:

$$\begin{aligned}\mathbb{S}(u,u)\frac{h(x)}{h^{(+)}(x)} &= \liminf_{n\to\infty}\frac{1}{n\xi(x)}K_n^{(+)}\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right)\frac{h\left(x+\frac{u}{n\xi(x)}\right)}{h^{(+)}\left(x+\frac{u}{n\xi(x)}\right)}\\ &\leq \liminf_{n\to\infty}\frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)}, x+\frac{u}{n\xi(x)}\right) \qquad (5.4.6)\end{aligned}$$

Taking the limit supremum for $n \to \infty$ in (5.4.3) and (5.4.4) gives

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n\xi(x)} K_n^{(-)}\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \frac{h\left(x + \frac{u}{n\xi(x)}\right)}{h^{(-)}\left(x + \frac{u}{n\xi(x)}\right)} \qquad (5.4.7) \\
&= \mathbb{S}(u,u)\frac{h(x)}{h^{(-)}(x)}
\end{aligned}$$

Consequently, combining (5.4.6) and (5.4.7) gives

$$\begin{aligned}
\mathbb{S}(u,u)\frac{h(x)}{h^{(+)}(x)} &\leq \liminf_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \leq \mathbb{S}(u,u)\frac{h(x)}{h^{(-)}(x)} \qquad (5.4.8)
\end{aligned}$$

Note that $|h^{(\pm)}(x) - h(x)| < \epsilon$, where $\epsilon > 0$. Consequently,

$$\frac{h(x)}{h^{(\pm)}(x)} = 1 + \mathcal{O}(\epsilon)$$

as $\epsilon \to 0$.
This means that, as $\epsilon \to 0$, (5.4.8) will become

$$\begin{aligned}
\mathbb{S}(u,u) &\leq \liminf_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) \leq \mathbb{S}(u,u)
\end{aligned}$$

from which we conclude that

$$\lim_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{u}{n\xi(x)}\right) = \mathbb{S}(u,u)$$

Proving uniform convergence is trivial at this point.

The off-diagonal case

Let $\epsilon > 0$. Let $x \in (-1,1)$. Let $u,\, v \in \mathbb{R}$. Let $K_n^{(+)}$, K_n, $w^{(+)}$ and w be as before for the diagonal case. Our aim is to prove that

$$\limsup_{n\to\infty} \left| \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) - \mathbb{S}(u,v) \right| = 0$$

where $\mathbb{S}(u,v)$ is the sine kernel.
By the triangle inequality,

$$\begin{aligned}
&\left|\frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right| \\
&\leq \frac{1}{n\xi(x)}\left|K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-K_n^{(+)}\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)\right| \\
&+\left|\frac{1}{n\xi(x)}K_n^{(+)}\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right|
\end{aligned} \tag{5.4.9}$$

Our theorem holds true for analytic weights, so particularly for $w^{(+)}$, we have that

$$\lim_{n\to\infty}\left|\frac{1}{n\xi(x)}K_n^{(+)}\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right|=0$$

Thus, (5.4.9) implies

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right| \qquad (5.4.10)\\
&\leq \limsup_{n\to\infty}\frac{1}{n\xi(x)}\left|K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-K_n^{(+)}\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)\right|
\end{aligned}$$

Let's focus on the right hand side of (5.4.10).
Lemma 5.3.2 tells us that

$$\begin{aligned}
|K_n(x,y)-K_n^{(+)}(x,y)| &\leq K_n(x,x)^{\frac{1}{2}}\left(K_n(y,y)-K_n^{(+)}(y,y)\frac{h(y)}{h^{(+)}(y)}\right)^{\frac{1}{2}} \\
&+\left|K_n^{(+)}(x,y)\right|\left|\frac{h(x)^{\frac{1}{2}}h(y)^{\frac{1}{2}}}{h^{(+)}(x)^{\frac{1}{2}}h^{(+)}(y)^{\frac{1}{2}}}-1\right|
\end{aligned} \tag{5.4.11}$$

At this point, we will do the following:

- Insert $x+\frac{u}{n\xi(x)}$ for x and $x+\frac{v}{n\xi(x)}$ for y into (5.4.11).
- Divide both sides of (5.4.11) by $n\xi(x)$.
- Then we will take the limit supremum for n going to infinity.

Using that Theorem 5.2.3 is already proven for the diagonal case and holds true for analytic weights, particularly for $w^{(+)}$, we can then deduce from (5.4.11) that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n\xi(x)}\left|K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-K_n^{(+)}\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)\right| \\
&\leq \mathbb{S}(u,u)^{\frac{1}{2}}\mathbb{S}(v,v)^{\frac{1}{2}}\left(1-\frac{h(x)}{h^{(+)}(x)}\right)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\left|\frac{h(x)}{h^{(+)}(x)}-1\right|
\end{aligned} \tag{5.4.12}$$

Using (5.4.12) with (5.4.10) gives

$$\limsup_{n\to\infty}\left|\frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right|$$
$$\leq \mathbb{S}(u,u)^{\frac{1}{2}}\mathbb{S}(v,v)^{\frac{1}{2}}\left(1-\frac{h(x)}{h^{(+)}(x)}\right)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\left|\frac{h(x)}{h^{(+)}(x)}-1\right|$$

As $|h^{(+)}(x)-h(x)|<\epsilon$ and $\epsilon>0$, we find that by letting $\epsilon\to 0$

$$\limsup_{n\to\infty}\frac{1}{n\xi(x)}\left|K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)-\mathbb{S}(u,v)\right|$$
$$\leq \mathbb{S}(u,u)^{\frac{1}{2}}\mathbb{S}(v,v)^{\frac{1}{2}}\,(0)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\cdot 0$$

which means that

$$\lim_{n\to\infty}\frac{1}{n\xi(x)}K_n\left(x+\frac{u}{n\xi(x)},x+\frac{v}{n\xi(x)}\right)=\frac{\sin\left(\pi(u-v)\right)}{\pi(u-v)}.$$

Proving uniform convergence is trivial at this point.

5.4.2 Proof of Theorem 5.2.4

The diagonal case

Let $\epsilon>0$. Let $K_n^{(\pm)}$, $w^{(\pm)}$, $h^{(\pm)}$ be as for the proof of Theorem 5.2.3. This time, we will set out to prove that for $u>0$

$$\lim_{n\to\infty}\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)=\mathbb{J}_\alpha(u,u)$$

Analogously to the proof for the diagonal case (see (5.4.2), (5.4.3) and (5.4.4)) of Theorem 5.2.3 we find that

$$\begin{aligned}&\frac{1}{2n^2}K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)\frac{h\left(1-\frac{u}{2n^2}\right)}{h^{(+)}\left(1-\frac{u}{2n^2}\right)}\\&\leq\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)\\&\leq\frac{1}{2n^2}K_n^{(-)}\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)\frac{h\left(1-\frac{u}{2n^2}\right)}{h^{(-)}\left(1-\frac{u}{2n^2}\right)}\end{aligned}\tag{5.4.13}$$

Theorem 5.2.4 holds true for analytic h by [46], so in particular for $h^{(\pm)}$ we have that

$$\lim_{n\to\infty}\frac{1}{2n^2}K_n^{(\pm)}\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)=\mathbb{J}_\alpha(u,u)$$

Hence taking the limit supremum for $n \to \infty$ in (5.4.13) gives

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \\
&\leq \lim_{n\to\infty} \frac{1}{2n^2} K_n^{(-)}\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \frac{h\left(1-\frac{u}{2n^2}\right)}{h^{(-)}\left(1-\frac{u}{2n^2}\right)} \\
&= \mathbb{J}_\alpha(u,u) \frac{h(1)}{h^{(-)}(1)}
\end{aligned} \tag{5.4.14}$$

In the same way, taking the limit infimum for $n \to \infty$ in (5.4.13) gives

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \\
&\geq \lim_{n\to\infty} \frac{1}{2n^2} K_n^{(+)}\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \frac{h\left(1-\frac{u}{2n^2}\right)}{h^{(+)}\left(1-\frac{u}{2n^2}\right)} \\
&= \mathbb{J}_\alpha(u,u) \frac{h(1)}{h^{(+)}(1)}
\end{aligned} \tag{5.4.15}$$

As $|h^{(\pm)}(1)-h(1)| < \epsilon$ and $\epsilon > 0$, we can conclude from (5.4.14) and (5.4.15) that for $\epsilon \to 0$,

$$\limsup_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \leq \mathbb{J}_\alpha(u,u)$$

and

$$\liminf_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) \geq \mathbb{J}_\alpha(u,u)$$

meaning that

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{u}{2n^2}\right) = \mathbb{J}_\alpha(u,u)$$

which concludes the first step of our proof. Proving uniform convergence is trivial at this point.

The off-diagonal case

The diagonal case behind us, we aim to tackle the off-diagonal case next: Let $\epsilon > 0$. Define $K_n^{(+)}$, $w^{(+)}$ as before in the diagonal case.
This time we will set out to show that for $u > 0$, $v > 0$

$$\limsup_{n\to\infty} \left| \frac{1}{2n^2} K_n\left(1-\frac{u}{2n^2}, 1-\frac{v}{2n^2}\right) - \mathbb{J}_\alpha(u,v) \right| = 0 \tag{5.4.16}$$

Using the triangle inequality, we see that

$$\begin{aligned}
&\left|\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-\mathbb{J}_\alpha(u,v)\right| \\
&\leq \left|K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)\right| \\
&+\left|\frac{1}{2n^2}K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-\mathbb{J}_\alpha(u,v)\right| \qquad (5.4.17)
\end{aligned}$$

As Theorem 5.2.4 holds true for analytic weights by [46], we see that particularly for $w^{(+)}$,

$$\limsup_{n\to\infty}\left|\frac{1}{2n^2}K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-\mathbb{J}_\alpha(u,v)\right|=0$$

So (5.4.17) implies

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-\mathbb{J}_\alpha(u,v)\right| \qquad (5.4.18)\\
&\leq \limsup_{n\to\infty}\frac{1}{2n^2}\left|K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)\right|
\end{aligned}$$

From Lemma 5.3.2 we learn that

$$\begin{aligned}
|K_n(x,y)-K_n^{(+)}(x,y)| \leq K_n(x,x)^{\frac{1}{2}}&\left(K_n(y,y)-K_n^{(+)}(y,y)\frac{h(y)}{h^{(+)}(y)}\right)^{\frac{1}{2}} \\
&+\left|K_n^{(+)}(x,y)\right|\left|\frac{h(x)^{\frac{1}{2}}h(y)^{\frac{1}{2}}}{h^{(+)}(x)^{\frac{1}{2}}h^{(+)}(y)^{\frac{1}{2}}}-1\right| \qquad (5.4.19)
\end{aligned}$$

We already know that

$$\lim_{n\to\infty}\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{u}{2n^2}\right)=\mathbb{J}_\alpha(u,u) \qquad (5.4.20)$$

by the proof of the diagonal case and

$$\lim_{n\to\infty}\frac{1}{2n^2}K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)=\mathbb{J}_\alpha(u,v) \qquad (5.4.21)$$

as $h^{(+)}$ is analytic for which Theorem 5.2.4 was proven in [46]. Combining (5.4.20) and (5.4.21) with (5.4.19), we find that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{2n^2}\left|K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)\right| \\
&\leq \mathbb{J}_\alpha(u,u)^{\frac{1}{2}}\left(\mathbb{J}_\alpha(v,v)-\mathbb{J}_\alpha(v,v)\frac{h(1)}{h^{(+)}(1)}\right)^{\frac{1}{2}} \\
&+|\mathbb{J}_\alpha(u,v)|\left|\frac{h(1)}{h^{(+)}(1)}-1\right| \qquad (5.4.22)
\end{aligned}$$

Extracting $\mathbb{J}_\alpha(v,v)$ out of the first term of the final expression of (5.4.22) we see that

$$\limsup_{n\to\infty} \frac{1}{2n^2}\left|K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-K_n^{(+)}\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)\right|$$
$$\leq \mathbb{J}_\alpha(u,u)\mathbb{J}_\alpha(v,v)\left(1-\frac{h(1)}{h^{(+)}(1)}\right)^{\frac{1}{2}}+|\mathbb{J}_\alpha(u,v)|\left|\frac{h(1)}{h^{(+)}(1)}-1\right|$$

Thus, (5.4.18) becomes

$$\limsup_{n\to\infty}\left|\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-J_\alpha(u,v)\right|$$
$$\leq \mathbb{J}_\alpha(u,u)\mathbb{J}_\alpha(v,v)\left(1-\frac{h(1)}{h^{(+)}(1)}\right)^{\frac{1}{2}}+|\mathbb{J}_\alpha(u,v)|\left|\frac{h(1)}{h^{(+)}(1)}-1\right| \qquad (5.4.23)$$

Again, as $|h^{(+)}(1)-h(1)|<\epsilon$ for $\epsilon>0$, (5.4.23) becomes, if $\epsilon\to 0$

$$\limsup_{n\to\infty}\left|\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)-J_\alpha(u,v)\right|$$
$$\leq \mathbb{J}_\alpha(u,u)\mathbb{J}_\alpha(v,v)\,(0)^{\frac{1}{2}}+|\mathbb{J}_\alpha(u,v)|\,|0|=0$$

So,

$$\lim_{n\to\infty}\frac{1}{2n^2}K_n\left(1-\frac{u}{2n^2},1-\frac{v}{2n^2}\right)=J_\alpha(u,v)$$

Proving uniform convergence is trivial at this point.

5.4.3 Proof of Theorem 5.2.5

The diagonal case

In this proof, we are dealing with weights $w(x)=h(x)\nu_{x_0}(x)w^{\alpha,\beta}(x)$ where $w^{\alpha,\beta}(x)$ is the Jacobi weight (see (5.2.1)) and $h(x)$ is a positive continuous function and

$$\nu_{x_0}(x)=\begin{cases}1 \text{ if } x<x_0\\ c^2 \text{ if } x\geq x_0\end{cases}$$

As in the proofs of Theorem 5.2.3 and Theorem 5.2.4, define positive polynomials $h^{(\pm)}$ such that

$$h^{(-)}(x)\leq h(x)\leq h^{(+)}(x) \text{ for } x\in[-1,1]$$

and $|h^{(\pm)}(x)-h(x)|<\epsilon$ for $\epsilon>0$ and weights

$$w^{(\pm)}(x)=h^{(\pm)}(x)\nu_{x_0}(x)w^{\alpha,\beta}(x).$$

Finally, let $K_n^{(\pm)}$ be the reproducing kernel with respect to $w^{(\pm)}$.
By Lemma 5.3.4

$$K_n^{(+)}(x,x)\frac{h(x)}{h^{(+)}(x)} \le K_n(x,x) \le K_n^{(-)}(x,x)\frac{h(x)}{h^{(-)}(x)} \tag{5.4.24}$$

Because of Theorem 2.1.1

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n^{(+)}\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,u) \tag{5.4.25}$$

and

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n^{(-)}\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,u) \tag{5.4.26}$$

Combining (5.4.24) and (5.4.25) we get

$$\mathbb{K}_c^{CHF}(u,u)\frac{h(x_0)}{h^{(+)}(x_0)} = \liminf_{n\to\infty} \frac{1}{n\xi(x_0)} K_n^{(+)}\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right)\frac{h(x_0)}{h^{(+)}(x_0)}$$

$$\le \liminf_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) \tag{5.4.27}$$

and combining (5.4.24) and (5.4.26) gives

$$\limsup_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) \tag{5.4.28}$$

$$\le \limsup_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right)\frac{h(x_0)}{h^{(-)}(x_0)} = \mathbb{K}_c^{CHF}(u,u)\frac{h(x_0)}{h^{(-)}(x_0)}$$

So

$$\mathbb{K}_c^{CHF}(u,u)\frac{h(x_0)}{h^{(+)}(x_0)} \le \liminf_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) \tag{5.4.29}$$

$$\le \limsup_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right)\frac{h(x_0)}{h^{(-)}(x_0)} \le \mathbb{K}_c^{CHF}(u,u)\frac{h(x_0)}{h^{(-)}(x_0)}$$

As $|h^{(\pm)}(x_0) - h(x_0)| < \epsilon$ for $\epsilon > 0$, we have that for $\epsilon \to 0$,

$$\frac{h(x_0)}{h^{\pm}(x_0)} = 1 + \mathcal{O}(\epsilon)$$

This means that if $\epsilon \to 0$, then (5.4.29) can be rewritten as

$$\mathbb{K}_c^{CHF}(u,u) \le \liminf_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) \tag{5.4.30}$$

$$\le \limsup_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right)\frac{h(x_0)}{h^{(-)}(x_0)} \le \mathbb{K}_c^{CHF}(u,u)$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0+\frac{u}{n\xi(x_0)}, x_0+\frac{u}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,u)$$

Proving uniform convergence is trivial at this point.

The off-diagonal case

Again, we proceed to the off-diagonal case: Let $K_n^{(+)}$, $w^{(+)}$ and $h^{(+)}$ be as before in the diagonal case. We will set out to prove that

$$\limsup_{n\to\infty}\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)-\mathbb{K}_c^{CHF}(u,u)\right|=0 \tag{5.4.31}$$

Note that by the triangle inequality the following holds for the left hand side of (5.4.31):

$$\begin{aligned}&\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)-\mathbb{K}_c^{CHF}(u,u)\right|\\ &\leq\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)\right.\\ &\left.-\frac{1}{n\xi(x_0)}K_n^{(+)}\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)\right|\\ &+\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)-\mathbb{K}_c^{CHF}(u,u)\right|\end{aligned} \tag{5.4.32}$$

As $h^{(+)}$ is analytic, we have by [32] that

$$\limsup_{n\to\infty}\left|\frac{1}{n\xi(x_0)}K_n^{(+)}\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)-\mathbb{K}_c^{CHF}(u,u)\right|=0$$

So (5.4.32) becomes

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)-\mathbb{K}_c^{CHF}(u,u)\right|\\ &\leq\limsup_{n\to\infty}\left|\frac{1}{n\xi(x_0)}K_n\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)\right.\\ &\qquad\left.-\frac{1}{n\xi(x_0)}K_n^{(+)}\left(x_0+\frac{u}{n\xi(x_0)},x_0+\frac{v}{n\xi(x_0)}\right)\right|\end{aligned} \tag{5.4.33}$$

Next, we will be focussing on finding an estimate of the right hand side of (5.4.33).
Because of Lemma 5.3.2 we have that

$$\begin{aligned}|K_n(x,y)-K_n^{(+)}(x,y)|&\leq K_n(x,x)^{\frac{1}{2}}\left(K_n(y,y)-K_n^{(+)}(y,y)\frac{h(y)}{h^{(+)}(y)}\right)\\ &+|K_n^{(+)}(x,y)|\left|\frac{h(x)^{\frac{1}{2}}h(y)^{\frac{1}{2}}}{h^{(+)}(x)^{\frac{1}{2}}h^{(+)}(y)^{\frac{1}{2}}}-1\right|\end{aligned} \tag{5.4.34}$$

Following the strategy of the proof of the off-diagonal case of Theorem 5.2.3 and Theorem 5.2.4, we insert $x_0+\frac{u}{n\xi(x_0)}$ for x, $x_0+\frac{v}{n\xi(x_0)}$ for y into (5.4.34), multiply both sides of (5.4.34) with $\frac{1}{n\xi(x_0)}$ and take the limit supremum for n going to infinity. This means that (5.4.34) will become

$$\limsup_{n\to\infty} \frac{1}{n\xi(x_0)} \left| K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) - K_n^{(+)} \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) \right|$$
$$\leq \mathbb{K}_c^{CHF}(u,u) \left(1 - \frac{h(x_0)}{h^{(+)}(x_0)} \right) + |\mathbb{K}_c^{CHF}(u,v)| \left| \frac{h(x_0)}{h^{(+)}(x_0)} - 1 \right|$$

as for the diagonal case the result already holds and our theorem is already proven for analytic weights by [32].

So (5.4.33) can be rewritten as

$$\limsup_{n\to\infty} \left| \frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) - \mathbb{K}_c^{CHF}(u,v) \right|$$
$$\leq \mathbb{K}_c^{CHF}(u,u) \left(1 - \frac{h(x_0)}{h^{(+)}(x_0)} \right) + |\mathbb{K}_c^{CHF}(u,v)| \left| \frac{h(x_0)}{h^{(+)}(x_0)} - 1 \right|$$

As $|h^{(+)}(x_0) - h(x_0)| < \epsilon$ for $\epsilon > 0$, we get that for $\epsilon \to 0$

$$\limsup_{n\to\infty} \left| \frac{1}{n\xi(x_0)} K_n \left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)} \right) - \mathbb{K}_c^{CHF}(u,v) \right|$$
$$\leq \mathbb{K}_c^{CHF}(u,u) \cdot 0 + |\mathbb{K}_c^{CHF}(u,v)| \, |0| = 0$$

which proves our theorem.
Proving uniform convergence is trivial at this point.

5.5 Limit behaviour for reproducing kernels with respect to a fixed, discontinuous weight

In this section, we will prove Theorem 5.2.6, Theorem 5.2.7 and Theorem 5.2.8 by repeating the strategy described in the previous section, except that this time we will be using continuous weights where we were using analytic weights before in the proofs of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5 in section 5.4. Specifically, we will split our proofs into two steps:

- The first step will be to obtain our result for the diagonal case, or $K_n(x,x)$. The main idea will be to find continuous weights $w^{(-)}(x)$, $w^{(+)}(x)$ that lie close to $w(x)$ within some specified interval and for which $w^{(-)}(x) \leq w(x) \leq w^{(+)}(x)$ for $x \in (-1,1)$. Then, taking appropriate limits, Lemma 5.3.4 will in every proof lead to the result for the diagonal case.

- The second step will then be to generalise the diagonal result to the off-diagonal result, using Lemma 5.3.2.

5.5.1 Proof of Theorem 5.2.6

Let $I = (a, b)$ be a subinterval of $[-1, 1]$, let $h(x)$ be continuous for $x \in I$ and let $0 < \delta << 1$. We want to find continuous weight functions $w^{(+)}$, $w^{(-)}$, for which

$$w^{(-)}(x) \leq w(x) \leq w^{(+)}(x) \text{ for } x \in [-1, 1]$$

with $w^{(\pm)}(x) = w(x)$ for $x \in [a + \delta, b - \delta]$. From that point onwards, we can simply repeat our strategy used in the proofs of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, except that this time, we do not need ϵ-arguments, as $w^{(\pm)}(x) = w(x)$ for $x \in [a + \delta, b - \delta]$.
We will start by constructing continuous functions $h^{(+)}$, $h^{(-)}$ related to $w^{(+)}$ and $w^{(-)}$ respectively in the sense that $w^{(\pm)}(x) = h^{(\pm)}(x)w^{\alpha,\beta}(x)$.
As $c \leq h(x) \leq d$ for $x \in [-1, 1]$ for some c, $d \in \mathbb{R}_{>0}$, we can define positive continuous functions $h^{(\pm)}(x)$ such that

$$h^{(\pm)}(x) = h(x) \text{ for } x \in [a + \delta, b - \delta] \tag{5.5.1}$$

as $h(x)$ is continuous for $x \in I$.
For $x \notin I$, we define

$$h^{(-)}(x) = c \text{ and } h^{(+)}(x) = d \tag{5.5.2}$$

What remains is to construct $h^{(\pm)}(x)$ for $x \in [a, a + \delta) \cup (b - \delta, b]$.
Let $l_1^{(-)}(x)$ be a polynomial for which

$$l_1^{(-)}(a) = c \text{ and } l_1^{(-)}(a + \delta) = h(a + \delta)$$

and let $l_2^{(-)}(x)$ be a polynomial for which

$$l_2^{(-)}(b - \delta) = h(b - \delta) \text{ and } l_2^{(-)}(b) = c$$

Similarly, let $l_1^{(+)}(x)$ be a polynomial for which

$$l_1^{(+)}(a) = d \text{ and } l_1^{(+)}(a + \delta) = h(a + \delta)$$

and let $l_2^{(+)}(x)$ be a polynomial for which

$$l_2^{(+)}(b - \delta) = h(b - \delta) \text{ and } l_2^{(+)}(b) = d$$

For $x \in [a, a + \delta)$ we define

$$h^{(-)} = \min\left\{l_1^{(-)}(x), h(x)\right\} \text{ and } h^{(+)}(x) = \max\left\{l_1^{(+)}(x), h(x)\right\} \tag{5.5.3}$$

For $x \in (b-\delta, b]$ we define

$$h^{(-)} = \min\left\{l_2^{(-)}(x), h(x)\right\} \text{ and } h^{(+)}(x) = \max\left\{l_2^{(+)}(x), h(x)\right\} \quad (5.5.4)$$

From (5.5.1), (5.5.2), (5.5.3) and (5.5.4) we can see that $h^{(-)}(x)$ and $h^{(+)}(x)$ are continuous for $x \in [-1, 1]$ and

$$c_1 \leq h^{(-)}(x) \leq h(x) \leq h^{(+)}(x) \leq c_2 \text{ for } x \in [-1, 1] \quad (5.5.5)$$

At this point one can proceed with $h^{(\pm)}(x)$ defined as in (5.5.1), (5.5.2), (5.5.3) and (5.5.4) almost exactly as before in the proof of Theorem 5.2.3 as was mentioned at the beginning of this proof.

5.5.2 Proof of Theorem 5.2.7

The strategy in this proof is essentially a repetition of the proof of Theorem 5.2.6.
Let $\delta > 0$, $[1-2\delta, 1] \subset [-1, 1]$.
We will start by constructing continuous functions $h^{(+)}$, $h^{(-)}$ related to $w^{(+)}$, $w^{(-)}$ in the sense that $w^{(\pm)}(x) = h^{(\pm)}(x) w^{\alpha,\beta}(x)$, for which $w(x) = w^{(\pm)}(x)$ for $x \in [1-\delta, 1]$ and

$$w^{(-)}(x) \leq w(x) \leq w^{(+)}(x)$$

for $x \in [-1, 1-\delta)$.
For $x \in [1-\delta, 1]$ we define

$$h^{(\pm)}(x) = h(x) \quad (5.5.6)$$

We know that $c \leq h(x) \leq d$ for $x \in [-1, 1]$. Define for $x \in [-1, 1-2\delta)$

$$h^{(-)}(x) = c \text{ and } h^{(+)}(x) = d \quad (5.5.7)$$

Lastly, we need to construct $h^{(\pm)}(x)$ for $x \in [1-2\delta, 1-\delta)$.
Let $l^{(-)}(x)$ be a polynomial for which

$$l^{(-)}(1-2\delta) = c \text{ and } l^{(-)}(1-\delta) = h(1-\delta)$$

and let $l^{(+)}(x)$ be a polynomial for which

$$l^{(+)}(1-2\delta) = d \text{ and } l^{(+)}(1-\delta) = h(1-\delta)$$

For $x \in [1-2\delta, 1-\delta)$ we define

$$h^{(-)}(x) = \min\left\{l^{(-)}(x), h(x)\right\} \text{ and } h^{(+)}(x) = \max\left\{l^{(+)}(x), h(x)\right\} \quad (5.5.8)$$

From (5.5.6), (5.5.7), (5.5.8) and the continuity of $h(x)$ for $x \in [1-2\delta, 1]$ we conclude that $h^{(+)}(x)$ and $h^{(-)}(x)$ are continuous for $x \in [-1, 1]$ and

$$h^{(-)}(x) \leq h(x) \leq h^{(+)}(x) \text{ for } x \in [-1, 1] \quad (5.5.9)$$

From hereon, we can simply repeat the proof of Theorem 5.2.4 with $h^{(\pm)}$ defined as in (5.5.6), (5.5.7) and (5.5.8).

5.5.3 Proof of Theorem 5.2.8

As was done in the proofs of Theorem 5.2.6 and Theorem 5.2.7, we aim to find continuous functions that approximate h the same way we used analytic functions to approximate h in the proofs of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5.
As was done in the proofs of Theorem 5.2.6 and Theorem 5.2.7, we will start by constructing continuous functions $h^{(-)}$, $h^{(+)}$ for which

$$c \le h^{(-)}(x) \le h(x) \le h^{(+)}(x) \le d$$

for $x \in [-1, 1]$.
Let $0 < \delta << 1$. If $h(x)$ is continuous for $x \in I$, $I \subset [-1, 1]$ an open subinterval and $x_0 \in I$, then for some δ we have that $h(x)$ is continuous for $x \in [x_0 - 2\delta, x_0 + 2\delta]$. Following the main idea of the proofs of Theorem 5.2.6 and Theorem 5.2.7, we define

$$h^{(\pm)}(x) = h(x) \text{ for } x \in [x_0 - \delta, x_0 + \delta] \tag{5.5.10}$$

For $x \in [-1, x_0 - \delta) \cup (x_0 + \delta, 1]$ we define

$$h^{(-)}(x) = c \text{ and } h^{(+)}(x) = d \tag{5.5.11}$$

Lastly, we need to construct $h^{(\pm)}(x)$ for $x \in [x_0 - 2\delta, x_0 - \delta) \cup (x_0 + \delta, x_0 + 2\delta]$.
Let $l_1^{(-)}(x)$ be a polynomial for which

$$l_1^{(-)}(x_0 - 2\delta) = c \text{ and } l_1^{(-)}(x_0 - \delta) = h(x_0 - \delta)$$

and let $l_1^{(+)}(x)$ be a polynomial for which

$$l_1^{(+)}(x_0 - 2\delta) = d \text{ and } l_1^{(+)}(x_0 - \delta) = h(x_0 - \delta)$$

For $x \in [x_0 - 2\delta, x_0 - \delta)$ we define

$$h^{(-)}(x) = \min\{l_1^{(-)}, h(x)\} \text{ and } h^{(+)}(x) = \max\{l_1^{(+)}, h(x)\} \tag{5.5.12}$$

Let $l_2^{(-)}(x)$ be a polynomial for which

$$l_2^{(-)}(x_0 + \delta) = h(x_0 + \delta) \text{ and } l_2^{(-)}(x_0 + \delta) = h(x_0 + \delta)$$

and let $l_2^{(+)}(x)$ be a polynomial for which

$$l_2^{(+)}(x_0 + \delta) = h(x_0 + \delta) \text{ and } l_2^{(+)}(x_0 + 2\delta) = d$$

For $x \in [x_0 + \delta, x_0 + 2\delta)$ we define

$$h^{(-)}(x) = \min\left\{l_2^{(-)}, h(x)\right\} \text{ and } h^{(+)}(x) = \max\left\{l_2^{(+)}, h(x)\right\} \tag{5.5.13}$$

From (5.5.10), (5.5.11), (5.5.12) and (5.5.13) we may conclude that $h^{(-)}(x)$ and $h^{(+)}(x)$ are continuous and

$$c \leq h^{(-)}(x) \leq h(x) \leq h^{(+)}(x) \leq d \text{ for } x \in [-1,1] \tag{5.5.14}$$

Repeating the proof of Theorem 5.2.5 with $h^{(\pm)}$ defined as in (5.5.10), (5.5.11), (5.5.12) and (5.5.13), will then give the desired result.

5.6 Limit behaviour for reproducing kernels with respect to a varying, continuous weight

5.6.1 Proof of Theorem 5.2.9 for analytic H

Before turning to our limit argument again, we first have to go through a Deift-Zhou steepest descent analysis once more:
In this section we will show how the Deift-Zhou steepest descent analysis for dealing with the case that

$$w_N(x) = e^{-NV(x)}$$

can be applied virtually unaltered when attacking the case that

$$w_N(x) = H(x)e^{-NV(x)}$$

where H is a positive real analytic function on $\operatorname{supp}\psi$. As an example, we will verify this statement for the one cut regular case, which we have already analysed in chapter 4.

The Riemann-Hilbert Problem

Based on (1.2.4) the Riemann-Hilbert problem to be analysed is

$$\begin{cases} Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & H(x)e^{-NV(x)} \\ 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R} \\ Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \text{ as } z \to \infty. \end{cases} \tag{5.6.1}$$

We shall now proceed with our Deift-Zhou steepest descent analysis, based on chapter 4.

The First Step: Transformation $Y \mapsto T$

Analogous to chapter 4, we will use the functions

$$g(z) = \int \log(z-s)\, d\mu_V(s) = \int \log(z-s)\psi_V(s)\, ds \tag{5.6.2}$$

and

$$\phi(z) = \pi \int_b^z ((s-b)(s-a))^{\frac{1}{2}}\, h(s)\, ds, \quad z \in \mathbb{C} \setminus (-\infty, b] \tag{5.6.3}$$

$$\tilde{\phi}(z) = \pi \int_a^z ((s-b)(s-a)))^{\frac{1}{2}}\, h(s)\, ds, \quad z \in \mathbb{C} \setminus [a, +\infty). \tag{5.6.4}$$

where μ_V is the equilibrium measure and ψ_V is its density function (see (5.2.9) and (5.2.10).
If we now put

$$T(z) = e^{n(l/2)\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-n(l/2)\sigma_3}, \tag{5.6.5}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the third Pauli matrix, then T satisfies the Riemann-Hilbert problem

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ T_+(x) = T_-(x) J_T(x) \text{ for } x \in \mathbb{R}, \\ T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty, \end{cases} \tag{5.6.6}$$

where

$$J_T(x) = \begin{cases} \begin{pmatrix} 1 & H(x) e^{-2n\tilde{\phi}(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x < a, \\ \begin{pmatrix} e^{2n\phi_+(x)} & H(x) \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} & \text{for } x \in (a, b), \\ \begin{pmatrix} 1 & H(x) e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x > b. \end{cases} \tag{5.6.7}$$

As before in chapter 4 the jump matrices for T on $(-\infty, a)$ and (b, ∞) tend to the identity matrix as $n \to \infty$.

The Second Step: Transformation $T \mapsto S$

The second transformation is, as was the first transformation, essentially the same as in chapter 4. We again split the jump on (a, b) as shown in Figure 5.1, where Σ_1 and Σ_2 are the same as in chapter 4. Recall that H is a positive real analytic function on $[a, b]$. So H has an analytic continuation to a region $U \subset \mathbb{C}$ that encloses $[a, b]$ on which Re $H > 0$. We choose the lens shaped region to lie within U.
We define S as follows:

- For z outside the lens, we put $S = T$.
- For z within the region enclosed by Σ_1 and (a, b),

$$S = T \begin{pmatrix} 1 & 0 \\ -H^{-1} e^{2n\phi} & 1 \end{pmatrix}. \tag{5.6.8}$$

- For z within the region enclosed by Σ_2 and (a, b),

$$S = T \begin{pmatrix} 1 & 0 \\ H^{-1}e^{2n\phi} & 1 \end{pmatrix}. \tag{5.6.9}$$

Then S satisfies the following Riemann-Hilbert problem:

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2) \\ S_+(z) = S_-(z)J_S(z) \text{ for } z \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \\ S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{5.6.10}$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ H(z)^{-1}e^{2n\phi(z)} & 1 \end{pmatrix} \text{ for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & H(z) \\ -H(z)^{-1} & 0 \end{pmatrix} \text{ for } z \in (a, b), \\ \begin{pmatrix} 1 & H(z)e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} \text{ for } z < a, \\ \begin{pmatrix} 1 & H(z)e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \text{ for } z > b, \end{cases} \tag{5.6.11}$$

The Third Step: Parametrix Away From Endpoints

The third step is the first part of the analysis that differs from the analysis in chapter 4. The Riemann-Hilbert problem to be solved in this step is

$$\begin{cases} N(z) \text{ is analytic in } \mathbb{C} \setminus [a, b] \\ N_+(x) = N_-(x) \begin{pmatrix} 0 & H(x) \\ -H(x)^{-1} & 0 \end{pmatrix} \text{ for } x \in (a, b) \\ N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{5.6.12}$$

So where before, in section 4.3.3, we were dealing with a jump

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

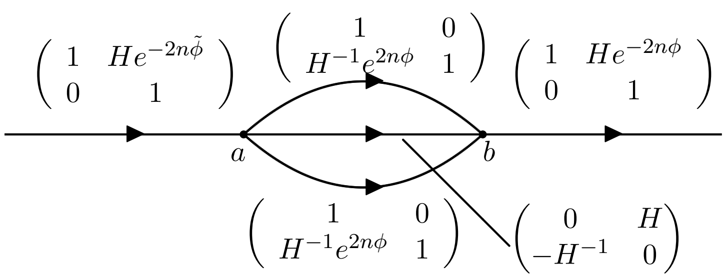

Figure 5.1: Jump matrices for S after opening of the lens

now we have a jump

$$\begin{pmatrix} 0 & H(x) \\ -H(x)^{-1} & 0 \end{pmatrix}$$

to deal with.
Let's refer to the parametrix away from the endpoints in section 4.3.3 as $\widehat{N}$. As was seen before in section 2.3.3 (or for example in [43] or [32]), we can easily find an adapted solution by defining

$$N(z) = D_{a,b,\infty}^{\sigma_3} \widehat{N}(z) D_{a,b}(z)^{-\sigma_3}$$

Here $D_{a,b}(z)$ is the Szegö function (see (1.1.5)) with respect to H and modified to have its jump behaviour on the interval $[a, b]$ instead of $[-1, 1]$, so

$$D_{a,b}(z) = e^{\frac{\sqrt{(z-a)(z-b)}}{2\pi} \int_a^b \frac{\log H(x) dx}{\sqrt{(b-x)(x-a)}(z-x)}}$$

(deducing $D_{a,b}(z)$ can be done in the same way as in Example 1.1.5) and $D_{a,b,\infty} = \lim_{z\to\infty} D_{a,b}(z)$.
Thus we have solved our Riemann-Hilbert problem for the parametrix away from the endpoints.

The Fourth Step: Parametrices Near Endpoints

As was done in section 4.3.4, the next step is to find the local parametrices close to the endpoints a and b. Near b, the local situation is described as in the left picture of Figure 4.2 with jump matrix

$$J_P(z) = J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ H(z)^{-1} e^{2n\phi(z)} & 1 \end{pmatrix} & \text{on } \Sigma_1 \cap U \text{ and } \Sigma_2 \cap U \\ \begin{pmatrix} 0 & H(z) \\ -H(z)^{-1} & 0 \end{pmatrix} & \text{on } (a, b) \cap U \\ \begin{pmatrix} 1 & H(z) e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{on } (b, \infty) \cap U \end{cases}$$

where U is a (small) disk around b and Σ_1 and Σ_2 are as in section 4.3.4.

We therefore want to find a matrix function P, that solves

$$\begin{cases} P(z) \text{ is analytic on } U \setminus (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \\ P_+(z) = P_-(z) J_P(z) \text{ on } (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U \\ P(z) = N(z) \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U \end{cases}$$

Note that $P(z)H(z)^{\frac{1}{2}\sigma_3}$ fulfills the jump conditions of the Riemann-Hilbert problem (4.3.16) in section 4.3.4, so all that remains is to find a suitable analytic prefactor E_n as before in section 4.3.4. If we choose

$$E_n(z) = \sqrt{\pi} N(z) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left(n^{2/3} f(z)\right)^{\sigma_3/4} \tag{5.6.13}$$

where N is the parametrix away from the endpoints as in the previous step of our steepest descent analysis, then the solution P must be

$$P(z) = E_n(z)\Phi(n^{\frac{2}{3}}f(z))e^{n\phi(z)\sigma_3}H(z)^{-\frac{1}{2}\sigma_3} \tag{5.6.14}$$

with $f(z)$ and Φ as in section 4.3.4. Recall from section 5.6.1 that for $z \in E$, we have that Re $H(z) > 0$. Thus, we choose U small enough to lie in E, which means that Re $H(z) > 0$ for $z \in U$, so we choose the square root of H along the negative axis.
As in section 4.3.4, a similar construction yields a parametrix $\tilde{P}$ in a small disc $\tilde{U}$ around a.

The Fifth Step: Transformation $S \mapsto R$

Again as in chapter 4, using the parametrices N, P, and $\tilde{P}$, we define the third transformation $S \mapsto R$ as follows

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ S(z)P(z)^{-1} & \text{for } z \in U \\ S(z)\tilde{P}(z)^{-1} & \text{for } z \in \tilde{U} \end{cases} \tag{5.6.15}$$

Then R has no jump on $[a,b] \setminus \overline{(U \cup \tilde{U})}$, as the jumps of S and N^{-1} cancel out. In U and $\tilde{U}$ the jumps of S cancel out with the jumps of P and $\tilde{P}$, leaving only jumps for R on the contour Σ_R shown in Figure 4.3.

The Riemann-Hilbert problem for R is

$$\begin{cases} R(z) \text{ is analytic on } \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z)J_R(z) \text{ for } z \in \Sigma_R \\ R(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases}$$

where

$$J_R(z) = \begin{cases} N(z)J_S(z)N(z)^{-1} \text{ for } z \in \Sigma_R \setminus (\partial U \cup \partial\tilde{U}) \\ P(z)N(z)^{-1} \text{ for } z \in \partial U \\ \tilde{P}(z)N(z)^{-1} \text{ for } z \in \partial\tilde{U} \end{cases}$$

with P, $\tilde{P}$ and N as in steps 3 and 4 of our analysis. By the same reasoning as in section 4.3.5, using that all jumps behave as $I + \mathcal{O}\left(\frac{1}{n}\right)$, we find that by the methods of [23], see also [43, Lemma 8.3], that

$$R(z) = I + \mathcal{O}\left(\frac{1}{n}\right) \tag{5.6.16}$$

Finding the limit behaviour of $K_{n,n}$

Using Proposition 1.2.1, we find:

$$\mathcal{K}_{n,n}(x,y) = \frac{1}{2\pi i(x-y)}(0,1)Y_+^{-1}(y)Y_+(x)\begin{pmatrix}1\\0\end{pmatrix} \tag{5.6.17}$$

where Y is as in section 5.6.1.
Next, using (5.6.5), we can rewrite (5.6.17) as

$$\begin{aligned}\mathcal{K}_{n,n}(x,y) = {}& \frac{1}{2\pi i(x-y)}(0,1)e^{-n(l/2)\sigma_3}e^{-ng_+(y)\sigma_3}T_+^{-1}(y)\\ & \cdot T_+(x)e^{ng_+(x)\sigma_3}e^{n(l/2)\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}\end{aligned} \tag{5.6.18}$$

In section 5.6.1 we learned that for z in the region enclosed by Σ_1 and (a,b), we now find

$$S = T\begin{pmatrix}1 & 0\\ -H^{-1}e^{2n\phi} & 1\end{pmatrix}$$

So (5.6.18) can be rewritten as

$$\begin{aligned}\mathcal{K}_{n,n}(x,y) = {}& \frac{1}{2\pi i(x-y)}(0,1)e^{-n(l/2)\sigma_3}e^{-ng_+(y)\sigma_3}\begin{pmatrix}1 & 0\\ -H(y)^{-1}e^{2n\phi_+(y)} & 1\end{pmatrix}S_+^{-1}(y)\\ & \cdot S_+(x)\begin{pmatrix}1 & 0\\ H(x)^{-1}e^{2n\phi_+(x)} & 1\end{pmatrix}e^{ng_+(x)\sigma_3}e^{n(l/2)\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}\end{aligned} \tag{5.6.19}$$

Using (4.3.2) with (4.3.3) and (4.3.4), we find that

$$2g_+(x) + 2\phi_+(x) + l = V(x)$$

This can be used to write (5.6.19) as

$$\begin{aligned}\mathcal{K}_{n,n}(x,y) = {}& \frac{1}{2\pi i(x-y)}(-1,1)e^{n\phi_+(y)\sigma_3}H(y)^{-\frac{1}{2}\sigma_3}S_+^{-1}(y)\\ & \cdot S_+(x)H(x)^{\frac{1}{2}\sigma_3}e^{-n\phi_+(x)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}H(x)^{-\frac{1}{2}}H(y)^{-\frac{1}{2}}e^{(n/2)(V(x)+V(y))}\end{aligned} \tag{5.6.20}$$

So multiplying both sides of (5.6.20) with

$$e^{-(n/2)(V(x)+V(y))}H(x)^{1/2}H(y)^{1/2}$$

then gives

$$\begin{aligned}K_{n,n}(x,y) = {}& \frac{1}{2\pi i(x-y)}(-1,1)e^{n\phi_+(y)\sigma_3}H(y)^{-\frac{1}{2}\sigma_3}S_+^{-1}(y)\\ & \cdot S_+(x)H(x)^{\frac{1}{2}\sigma_3}e^{-n\phi_+(x)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}\end{aligned} \tag{5.6.21}$$

Due to (5.6.15) we have that

$$H(y)^{-\frac{1}{2}\sigma_3}S_+(y)^{-1}S_+(x)H(x)^{\frac{1}{2}\sigma_3} = \begin{cases} H(y)^{-\frac{1}{2}\sigma_3}N_+(y)^{-1}R(y)^{-1}R(x)N_+(x)H(x)^{\frac{1}{2}\sigma_3} \\ \text{for } x, y \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ H(y)^{-\frac{1}{2}\sigma_3}P_+(y)^{-1}R(y)^{-1}R(x)P_+(x)H(x)^{\frac{1}{2}\sigma_3} \\ \text{for } x, y \in U \\ H(y)^{-\frac{1}{2}\sigma_3}\tilde{P}_+(y)^{-1}R(y)^{-1}R(x)\tilde{P}_+(x)H(x)^{\frac{1}{2}\sigma_3} \\ \text{for } x, y \in \tilde{U} \end{cases} \tag{5.6.22}$$

Recall that (5.6.16) tells us that

$$R(z) = I + \mathcal{O}\left(\frac{1}{n}\right)$$

leading to

$$H(y)^{-\frac{1}{2}\sigma_3}S_+(y)^{-1}S_+(x)H(x)^{\frac{1}{2}\sigma_3} = \begin{cases} H(y)^{-\frac{1}{2}\sigma_3}N_+(y)^{-1}N_+(x)H(x)^{\frac{1}{2}\sigma_3} + \mathcal{O}\left(\frac{1}{n}\right) \\ \text{for } x, y \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ H(y)^{-\frac{1}{2}\sigma_3}P_+(y)^{-1}P_+(x)H(x)^{\frac{1}{2}\sigma_3} + \mathcal{O}\left(\frac{1}{n}\right) \\ \text{for } x, y \in U \\ H(y)^{-\frac{1}{2}\sigma_3}\tilde{P}_+(y)^{-1}\tilde{P}_+(x)H(x)^{\frac{1}{2}\sigma_3} + \mathcal{O}\left(\frac{1}{n}\right) \\ \text{for } x, y \in \tilde{U} \end{cases} \tag{5.6.23}$$

At this point, we will be focussing on the behaviour close to b and the behaviour away from the endpoints. The behaviour around a can be deduced in the same way as around b.

For x, y away from the endpoints, expand

$$N_+(x)H(x)^{\frac{1}{2}\sigma_3}$$

as a power series around y, resulting in

$$N_+(x)H(x)^{\frac{1}{2}\sigma_3} = N_+(y)H(y)^{\frac{1}{2}\sigma_3} + \mathcal{O}(x-y)$$

Thus,

$$\begin{aligned} H(y)^{-\frac{1}{2}\sigma_3}S_+(y)^{-1}S_+(x)H(x)^{\frac{1}{2}\sigma_3} &= H(y)^{-\frac{1}{2}\sigma_3}N_+(y)^{-1}N_+(x)H(x)^{\frac{1}{2}\sigma_3} \\ &= I + \mathcal{O}(x-y) \end{aligned} \tag{5.6.24}$$

as N and thus $N(y)H(y)^{\frac{1}{2}\sigma_3}$ is bounded as long as y stays away from the endpoints.

For z close to b, we have that, using (5.6.14)

$$\begin{aligned} H(y)^{-\frac{1}{2}\sigma_3} S_+(y)^{-1} S_+(x) H(x)^{\frac{1}{2}\sigma_3} &= H(y)^{-\frac{1}{2}\sigma_3} P_+(y)^{-1} P_+(x) H(x)^{\frac{1}{2}\sigma_3} \\ &= e^{-n\phi(z)\sigma_3} \Phi(n^{\frac{2}{3}} f(y))^{-1} E_n(y)^{-1} \\ &\cdot E_n(x) \Phi(n^{\frac{2}{3}} f(x)) e^{n\phi(x)\sigma_3} \end{aligned} \tag{5.6.25}$$

Expanding $E_n(x)$ as a power series around y gives

$$E_n(x) = E_n(y) + \mathcal{O}(x-y)$$

and therefore

$$E_n(y)^{-1} E_n(x) = I + \mathcal{O}(x-y)$$

Thus, (5.6.25) becomes

$$\begin{aligned} H(y)^{-\frac{1}{2}\sigma_3} S_+(y)^{-1} S_+(x) H(x)^{\frac{1}{2}\sigma_3} &= e^{-n\phi(z)\sigma_3} \Phi(n^{\frac{2}{3}} f(y))^{-1} \Phi(n^{\frac{2}{3}} f(x)) e^{n\phi(x)\sigma_3} \\ &+ \mathcal{O}(x-y) \end{aligned} \tag{5.6.26}$$

So for x, y away from the endpoints, we have by (5.6.24), (5.6.21) and (5.6.26) that

$$K_{n,n}(x,y) = \frac{1}{2\pi i(x-y)} (-e^{n\phi_+(y)}, e^{-n\phi_+(y)}) \begin{pmatrix} e^{-n\phi_+(x)} \\ e^{n\phi_+(x)} \end{pmatrix} + \mathcal{O}(1)$$

which reduces to

$$K_{n,n}(x,y) = \frac{1}{2\pi i(x-y)} \left(e^{n(\phi_+(x)-\phi_+(y))} - e^{-n(\phi_+(x)-\phi_+(y))} \right) + \mathcal{O}(1)$$

Note that the right hand side of this last equation is independent of H, so that

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{v}{n\psi_V(x)} \right) = \frac{\sin \pi(u-v)}{\pi(u-v)}$$

as it was already proven for $H = 1$ in [23].
Next, let's prove the limit behaviour at the boundary point b:
By (5.6.25) and (5.6.21)

$$K_{n,n}(x,y) = \frac{1}{2\pi i(x-y)} (-1,1) \Psi_+^{-1}(n^{\frac{2}{3}} f(y)) \Psi_+(n^{\frac{2}{3}} f(x)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathcal{O}(1) \tag{5.6.27}$$

Note that the right hand side of (5.6.27) is independent of H, so again we invoke [23] to conclude that

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b + \frac{u}{(cn)^{2/3}}, b + \frac{v}{(cn)^{2/3}} \right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

5.6.2 Proof of Theorem 5.2.9 for continuous H

Analogously to the proofs of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, we will separate our proof again into two parts, the first being the proof for the diagonal case, the second being the proof for the off-diagonal case.

The diagonal case

Let $\epsilon > 0$. By the Weierstrass Approximation Theorem, we can find positive polynomials $H^{(+)}(z)$, $H^{(-)}(z)$ such that

$$H^{(-)}(x) \leq H(x) \leq H^{(+)}(x)$$

and

$$|H(x) - H^{(\pm)}(x)| < \epsilon \text{ for } x \in [a,b]$$

Furthermore, let $K_{n,N}^{(\pm)}$ (see (5.2.8)) be the normalised reproducing kernel with respect to the weight $w_N^{(\pm)}(x) = H^{(\pm)}(x)e^{-NV(x)}$.
By Lemma 5.3.4

$$K_{n,n}^{(+)}(x,x)\frac{H(x)}{H^{(+)}(x)} \leq K_{n,n}(x,x) \leq K_{n,n}^{(-)}(x,x)\frac{H(x)}{H^{(-)}(x)} \tag{5.6.28}$$

In the same way as was done in the proof of Theorem 5.2.3, this time because Theorem 5.2.9 has already been proven for analytic H in the previous section, we find that for $x \in (a,b)$ and $u \in \mathbb{R}$

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}^{(+)}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{u}{n\psi_V(x)}\right) \frac{H\left(x + \frac{u}{n\psi_V(x)}\right)}{H^{(+)}\left(x + \frac{u}{n\psi_V(x)}\right)} \\ = \mathbb{S}(u,u)\frac{H(x)}{H^{(+)}(x)}\end{aligned} \tag{5.6.29}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}^{(-)}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{u}{n\psi_V(x)}\right) \frac{H\left(x + \frac{u}{n\psi_V(x)}\right)}{H^{(-)}\left(x + \frac{u}{n\psi_V(x)}\right)} \\ = \mathbb{S}(u,u)\frac{H(x)}{H^{(-)}(x)}\end{aligned} \tag{5.6.30}$$

Here

$$\mathbb{S}(u,v) = \frac{\sin\left(\pi(u-v)\right)}{\pi(u-v)}$$

as in the proof of Theorem 5.2.3.
So combining (5.6.28), (5.6.29) and (5.6.30) gives us

$$\mathbb{S}(u,u)\frac{H(x)}{H^{(+)}(x)} \leq \liminf_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x+\frac{u}{n\psi_V(x)}, x+\frac{u}{n\psi_V(x)}\right)$$
$$\leq \limsup_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x+\frac{u}{n\psi_V(x)}, x+\frac{u}{n\psi_V(x)}\right) \leq \mathbb{S}(u,u)\frac{H(x)}{H^{(-)}(x)}$$

and as both

$$\frac{H(x)}{H^{(+)}(x)} = 1+\mathcal{O}(\epsilon) \text{ and } \frac{H(x)}{H^{(-)}(x)} = 1+\mathcal{O}(\epsilon) \text{ as } \epsilon\to 0$$

we find that for $\epsilon \to 0$

$$\mathbb{S}(u,u) \leq \liminf_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x+\frac{u}{n\psi_V(x)}, x+\frac{u}{n\psi_V(x)}\right)$$
$$\leq \limsup_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x+\frac{u}{n\psi_V(x)}, x+\frac{u}{n\psi_V(x)}\right) \leq \mathbb{S}(u,u)$$

so

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x+\frac{u}{n\psi_V(x)}, x+\frac{u}{n\psi_V(x)}\right) = \mathbb{S}(u,u)$$

which completes the proof regarding the sine kernel.
Next we will proceed to the Airy kernel:
Using (5.6.28) once more, we get, because of the fact that Theorem 5.2.9 has already been proven for analytic H in the previous section,

$$\liminf_{n\to\infty} \frac{1}{(cn)^{2/3}} K^{(+)}_{n,n}\left(b+\frac{u}{(cn)^{2/3}}, b+\frac{u}{(cn)^{2/3}}\right) \frac{H\left(b+\frac{u}{(cn)^{2/3}}\right)}{H^{(+)}\left(b+\frac{u}{(cn)^{2/3}}\right)}$$
$$= \mathbb{A}(u,u)\frac{H(b)}{H^{(+)}(b)} \quad (5.6.31)$$

and

$$\limsup_{n\to\infty} \frac{1}{(cn)^{2/3}} K^{(-)}_{n,n}\left(b+\frac{u}{(cn)^{2/3}}, b+\frac{u}{(cn)^{2/3}}\right) \frac{H\left(b+\frac{u}{(cn)^{2/3}}\right)}{H^{(-)}\left(b+\frac{u}{(cn)^{2/3}}\right)}$$
$$= \mathbb{A}(u,u)\frac{H(b)}{H^{(-)}(b)} \quad (5.6.32)$$

where

$$\mathbb{A}(u,v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

Combining (5.6.28), (5.6.31) and (5.6.32), we get

$$\mathbb{A}(u,u)\frac{H(b)}{H^{(+)}(b)} \leq \liminf_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b+\frac{u}{(cn)^{2/3}}, b+\frac{u}{(cn)^{2/3}}\right)$$
$$\leq \limsup_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b+\frac{u}{(cn)^{2/3}}, b+\frac{u}{(cn)^{2/3}}\right) \leq \mathbb{A}(u,u)\frac{H(b)}{H^{(-)}(b)}$$

Because both

$$\frac{H(b)}{H^{(+)}(b)} = 1+\mathcal{O}(\epsilon) \text{ and } \frac{H(b)}{H^{(+)}(b)} = 1+\mathcal{O}(\epsilon) \text{ as } \epsilon\to 0$$

we can conclude that, for $\epsilon \to 0$,

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b+\frac{u}{(cn)^{2/3}}, b+\frac{v}{(cn)^{2/3}}\right) = \mathbb{A}(u,u)$$

The off-diagonal case

Let $\epsilon > 0$. Let $H^{(+)}$, $K_{n,n}^{(+)}$ be as before in the proof of the diagonal case. As was the case in the proofs of Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, our main tool will be Lemma 5.3.2, which tells us that

$$\begin{aligned} &|K_{n,n}(x,y) - K_{n,n}^{(+)}(x,y)| \\ &\leq K_{n,n}(x,x)^{\frac{1}{2}}\left(K_{n,n}(y,y) - K_{n,n}^{(+)}(y,y)\frac{H(y)}{H^{(+)}(y)}\right)^{\frac{1}{2}} \\ &+ \left|K_{n,n}^{(+)}(x,y)\right| \left|\frac{H(x)^{\frac{1}{2}}H(y)^{\frac{1}{2}}}{H^{(+)}(x)^{\frac{1}{2}}H^{(+)}(y)^{\frac{1}{2}}} - 1\right| \end{aligned} \tag{5.6.33}$$

First we will prove the case for the sine kernel:
Let u, $v \in \mathbb{R}$ and $x \in (a,b)$. Before continuing, we need some new notation, to prevent formulas of becoming to large later on. Let

$$x_n = x + \frac{u}{n\psi_V(x)}$$

and

$$y_n = x + \frac{v}{n\psi_V(x)}$$

We will show that

$$\limsup_{n\to\infty} \left|\frac{1}{n\psi_V(x)} K_{n,n}(x_n,y_n) - \mathbb{S}(u,v)\right| = 0 \tag{5.6.34}$$

The triangle inequality tells us that

$$\begin{aligned}&\left|\frac{1}{n\psi_V(x)}K_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|\\&\leq\frac{1}{n\psi_V(x)}|K_{n,n}(x_n,y_n)-K^{(+)}_{n,n}(x_n,y_n)|\\&+\left|\frac{1}{n\psi_V(x)}K^{(+)}_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|\end{aligned}\tag{5.6.35}$$

As $K^{(+)}_{n,n}$ already has the desired limit behaviour because of the proof of Theorem 5.2.9 for analytic H in the previous section, due to $H^{(+)}$ being analytic, we see that

$$\limsup_{n\to\infty}\left|\frac{1}{n\psi_V(x)}K^{(+)}_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|=0$$

Thus, (5.6.35) gives us

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{n\psi_V(x)}K_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|\\&\leq\limsup_{n\to\infty}\frac{1}{n\psi_V(x)}\left|K_{n,n}(x_n,y_n)-K^{(+)}_{n,n}(x_n,y_n)\right|\end{aligned}\tag{5.6.36}$$

Using (5.6.33), noticing that for both the diagonal case and the case that H is analytic we have already proved our theorem, we find that

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n\psi_V(x)}|K_{n,n}(x_n,y_n)-K^{(+)}_{n,n}(x_n,y_n)|\\&\leq\mathbb{S}(u,u)^{\frac{1}{2}}\left(\mathbb{S}(v,v)-\mathbb{S}(v,v)\frac{H(x)}{H^{(+)}(x)}\right)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\left|\frac{H(x)}{H^{(+)}(x)}-1\right|\end{aligned}\tag{5.6.37}$$

Hence, using (5.6.36) with (5.6.37), we see that

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{n\psi_V(x)}K_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|\\&\leq\mathbb{S}(u,u)^{\frac{1}{2}}\left(\mathbb{S}(v,v)-\mathbb{S}(v,v)\frac{H(x)}{H^{(+)}(x)}\right)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\left|\frac{H(x)}{H^{(+)}(x)}-1\right|\end{aligned}$$

Because H and $H^{(+)}$ can be chosen to be arbitrarily close to each other, we get

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{n\psi_V(x)}|K_{n,n}(x_n,y_n)-\mathbb{S}(u,v)\right|\\&\leq\mathbb{S}(u,u)^{\frac{1}{2}}\mathbb{S}(v,v)^{\frac{1}{2}}\,(0)^{\frac{1}{2}}+|\mathbb{S}(u,v)|\,|0|=0\end{aligned}\tag{5.6.38}$$

Secondly, we will prove the case for the Airy kernel, by showing that, writing

$$\widehat{x_n} = b + \frac{u}{(cn)^{2/3}}$$

and

$$\widehat{y_n} = b + \frac{v}{(cn)^{2/3}}$$

that

$$\limsup_{n\to\infty} \left| \frac{1}{(cn)^{2/3}} K_{n,n}(\widehat{x_n}, \widehat{y_n}) - \mathbb{A}(u,v) \right| = 0 \tag{5.6.39}$$

Analogously to the sine kernel case described before, we find that

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{1}{(cn)^{2/3}} K_{n,n}(\widehat{x_n}, \widehat{y_n}) - \mathbb{A}(u,v) \right| \\ &\leq \limsup_{n\to\infty} \frac{1}{(cn)^{2/3}} |K_{n,n}(\widehat{x_n}, \widehat{y_n}) - K_{n,n}^{(+)}(\widehat{x_n}, \widehat{y_n})| \end{aligned} \tag{5.6.40}$$

Using that the kernel behaviour around b has already been deduced for the diagonal case and for analytic H, we learn from (5.6.33) that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{(cn)^{2/3}} |K_{n,n}(\widehat{x_n}, \widehat{y_n}) - K_{n,n}^{(+)}(\widehat{x_n}, \widehat{y_n})| \\ &\leq \mathbb{A}(u,u)^{\frac{1}{2}} \left(\mathbb{A}(v,v) - \mathbb{A}(v,v) \frac{H(b)}{H^{(+)}(b)} \right)^{\frac{1}{2}} + |\mathbb{A}(u,v)| \left| \frac{H(b)}{H^{(+)}(b)} - 1 \right| \end{aligned} \tag{5.6.41}$$

Combining (5.6.40) and (5.6.41), we get

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{1}{(cn)^{2/3}} |K_{n,n}(\widehat{x_n}, \widehat{y_n}) - \mathbb{A}(u,v) \right| \\ &\leq \mathbb{A}(u,u)^{\frac{1}{2}} \left(\mathbb{A}(v,v) - \mathbb{A}(v,v) \frac{H(b)}{H^{(+)}(b)} \right)^{\frac{1}{2}} + |\mathbb{A}(u,v)| \left| \frac{H(b)}{H^{(+)}(b)} - 1 \right| \end{aligned}$$

which leads, for the same reasons as (5.6.37), to

$$\limsup_{n\to\infty} \left| \frac{1}{(cn)^{2/3}} K_{n,n}(\widehat{x_n}, \widehat{y_n}) - \mathbb{A}(u,v) \right| = 0$$

Chapter 6

Conclusion and outlook for future research

The aim of this dissertation was to further our understanding and generalise results on recurrence coefficients and reproducing kernels obtained through Deift-Zhou steepest descent analysis. We have succeeded in this respect in the following sense:

- We have verified transitions between the sine kernel, the Bessel kernel $\mathbb{J}_\alpha$, the Bessel kernel $\mathbb{J}_\alpha^0$ and the Confluent Hypergeometric kernel $\mathbb{K}_c^{CHF}$ (see Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3).
- We have obtained asymptotics for the recurrence coefficients with respect to certain varying weights through Deift-Zhou steepest descent analysis and shown how for one of the respective asymptotic series we can in fact conclude that an infinite amount of terms cancel out, straight from the steepest descent analysis (see Theorem 4.1.4).
- We have generalised a number of universality results for normalised reproducing kernels with respect to (piecewise) analytic weights to the case that the weight function is continuous on some subset of its support (see Theorem 5.2.6, Theorem 5.2.7, Theorem 5.2.8 and Theorem 5.2.9).

Several avenues of research related to topics discussed in this thesis remain unexplored:

- It should be possible to apply the techniques used in chapter 3 to other kernels that do not have explicit formulas as well.
- In chapter 5 we generalised several results regarding limit behaviour of normalised reproducing kernels with respect to (piecewise) analytic weights to the case that the weight function was only continuous on a subset of its support. What can be said about limit behaviour of

recurrence coefficients, orthogonal polynomials etc.?
Also, in chapter 5 we still work with real analytic functions V in our varying weight functions w_N. The techniques we have used in chapter 5 will have to be adjusted (probably only slightly) to be able to weaken the assumptions on V from analyticity to continuity. It might also be possible to approximate by a suitable analytic weight, e.g. coming from a Fourier expansion and then doing a Riemann-Hilbert analysis in which the lens is opened with a width that is decreasing as n is increasing.

Chapter 7

Nederlandse samenvatting (Dutch summary)

In dit proefschrift bespreken we resultaten gerelateerd aan orthogonale veeltermen en in het bijzonder de asymptotiek van recurrentie coëfficiënten en reproducerende kernen. We definiëren orthogonale veeltermen p_n met graad $\deg p_n = n$ middels een Borel maat μ met drager $\Omega \subset \mathbb{R}$ en de eigenschap dat

$$\int_\Omega p_i(x)p_j(x)d\mu(x) = 0 \text{ als } i \neq j$$
$$\int_\Omega p_i(x)p_j(x)d\mu(x) \neq 0 \text{ als } i = j$$

We beperken ons tot het geval dat $d\mu$ weergegeven kan worden als $d\mu(x) = w(x)dx$, waarbij we $w(x) \geq 0$ voor $x \in \Omega$ de *gewichtsfunctie* noemen. In dit proefschrift is Ω gelijk aan het interval $[-1, 1]$, dan wel $\mathbb{R}$.
We spreken van *orthonormale veeltermen* $\{p_n\}_{n=0}^\infty$ als

$$\int_\Omega p_i(x)p_j(x)d\mu(x) = \begin{cases} 0 \text{ als } i \neq j \\ 1 \text{ als } i = j \end{cases}$$

We spreken van *monisch orthogonale veeltermen* $\{\pi_n\}_{n=0}^\infty$ als de veeltermen $\{\pi_n\}_{n=0}^\infty$ orthogonaal zijn en kopcoëfficiënt gelijk aan 1 hebben.
Met recurrentie coëfficiënten a_n, b_n bedoelen we de coëfficiënten in de volgende recurrente betrekking

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x) \tag{7.0.1}$$

(zie [64]).
Met een reproducerende kern, of beter, een *genormaliseerde* reproducerende

kern, bedoelen we een functie

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(y)^{\frac{1}{2}}$$

Als de orthogonaliteit gedefinieerd is met betrekking tot een variërend gewicht, oftewel een gewicht afhankelijk van een parameter N, passen we onze notatie aan door een onderschrift N toe te voegen aan alle relevante symbolen. Dus w wordt w_N, p_n wordt $p_{n,N}$, π_n wordt $\pi_{n,N}$, a_n wordt $a_{n,N}$, b_n wordt $b_{n,N}$ en K_n wordt $K_{n,N}$.
De theorie van orthogonale veeltermen kent tal van toepassingen in evenzoveel gebieden: Om een paar voorbeelden te noemen: Statistische fysica (zie onder andere [8]), aërodynamica (zie bijvoorbeeld [7]), random matrices (zie onder meer [14], [15] en [19]) en differentievergelijkingen (zie bijvoorbeeld [12]).
Iedere techniek die informatie geeft over het gedrag van orthogonale veeltermen met betrekking tot een algemeen gewicht is daarom van het hoogste belang.
En hier is een rol weggelegd voor de Deift-Zhou steilste afdalingsmethode (zie bijvoorbeeld [15] en [19]).
De Deift-Zhou steilste afdalingsmethode is een techniek die ons in staat stelt asymptotisch gedrag van orthogonale veeltermen te vinden via het toepassen van transformaties op een gerelateerd Riemann-Hilbert probleem.
Ruwweg doet de Deift-Zhou steilste afdalingsmethode het volgende: Gegeven veeltermen $\{p_n\}_{n=0}^{\infty}$ die orthogonaal zijn met betrekking tot een zeker analytisch gewicht w, dan stelt de Deift-Zhou steilste afdalingsmethode ons in staat het asymptotisch gedrag van de veeltermen en gerelateerde objecten zoals recurrentie coëfficiënten en reproducerende kernen te bepalen.
De klemtoon van dit proefschrift ligt bij het (waar mogelijk) generaliseren en verder verfijnen van resultaten verkregen met deze methode.
Concreet:

7.1 In hoofdstuk 1 en hoofdstuk 2:

Hoofdstuk 1 en hoofdstuk 2 fungeren hoofdzakelijk als een overzicht voor Riemann-Hilbert problemen, gerelateerde technieken en de Deift-Zhou steilste afdalingsmethode. In hoofdstuk 1 en hoofdstuk 2 bevinden zich dus geen nieuwe resultaten.

7.2 In hoofdstuk 3:

In hoofdstuk 3 laten we zien hoe middels de Deift-Zhou steilste afdalingsmethode relaties tussen verschillend limietgedrag voor reproducerende kernen

afgeleid kunnen worden. De techniek is gebaseerd op [42] en [41].
We bestuderen de volgende kernen:

- De *sinus kern*

$$\frac{\sin(\pi(x-y))}{\pi(x-y)}$$

- De *Bessel kern*

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'(\sqrt{x})}{2(x-y)}$$

waarbij J_α de Bessel functie van het eerste type van orde α is.

- Een tweede Bessel kern

$$\mathbb{J}^0_\alpha(x,y) = \pi\left(\frac{|x|}{x}\right)^\alpha\left(\frac{|y|}{y}\right)^\alpha\sqrt{x}\sqrt{y}\frac{J_{\alpha+\frac{1}{2}}(\pi x)J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x)J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)}$$

waarbij $\alpha > -\frac{1}{2}$ en $J_{\alpha\pm\frac{1}{2}}$ de Bessel functie van het eerste type van orde $\alpha \pm \frac{1}{2}$ is (zie [3], [34] en opmerking 1.2 van [47]). Verder hebben alle functies die voor komen in de uitdrukking voor $\mathbb{J}^0_\alpha$ een snede langs de negatieve reële rechte (waar van toepassing). Voor negatieve waarden van x schrijven we $x^\alpha = e^{\alpha\pi i}|x|^\alpha$ en $\sqrt{x} = e^{\frac{1}{2}\pi i}\sqrt{|x|}$.

- De *Confluent Hypergeometrische kern*

$$\mathbb{K}^{CHF}_c(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda;2\pi ix);G(\lambda;2\pi iy)]$$

waarbij $\lambda = \frac{i\log c}{\pi}$, $G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$, met $\phi(a,c;z)$ de confluente hypergeometrische functie van het eerste type en
$[f(x);g(y)] = f(x)g(y) - f(y)g(x)$.

We bewijzen dat

Stelling 7.2.1. Voor $s > 0$, voor alle $x, y \in \mathbb{R}$ en $\alpha > -1$,

$$2\pi s\mathbb{J}_\alpha\left(s^2+2\pi xs, s^2+2\pi ys\right) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

voor $s \to \infty$.

Stelling 7.2.2. Voor $s \in \mathbb{R}$, voor alle $x, y \in \mathbb{R}$ en $c > 0$,

$$\mathbb{K}^{CHF}_c(x+s, y+s) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

voor $s \to \pm\infty$.

Stelling 7.2.3. Voor $s \in \mathbb{R}$, voor alle $x, y \in \mathbb{R}$,

$$\mathbb{J}^0_\alpha (s+x, s+y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right)$$

voor $s \to \pm\infty$.

Het dient opgemerkt te worden, dat hoewel dit nieuwe resultaten zijn, de resultaten voor de Bessel kernen langs eenvoudiger weg gevonden kunnen worden, door rechtstreeks naar de asymptotiek van de Bessel functie te kijken. Het belang van dit hoofdstuk ligt bij de gebruikte methode, die ook van toepassing zou moeten zijn als er voor de betreffende kern geen expliciete uitdrukking bestaat.

7.3 In hoofdstuk 4:

In hoofdstuk 4 onderzoeken we het asymptotisch gedrag van de recurrentie coëfficiënten $a_{n,N}$ en $b_{n,N}$ in de recurrentie relatie

$$x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$$

voor monisch orthogonale veeltermen met betrekking tot variërende exponentiële gewichten

$$w_N(x) = e^{-NV(x)}$$

Hier is V reëel analytisch en

$$\frac{V(x)}{\log(1+x^2)} \to \infty \text{ voor } x \to \pm\infty$$

Verder bestaat er een zogeheten evenwichtsmaat $d\mu_V = \psi_V(x)dx$ gerelateerd aan V met compacte drager het interval $[a, b]$ waar het de volgende vorm heeft:

$$\psi_V(x)\,dx = \sqrt{(b-x)(x-a)}h(x)\chi_{[a,b]}(x)\,dx$$

Hier is h reëel analytisch, strikt positief op $[a, b]$ en

$$2\int \log|x-y|^{-1}\,d\mu_V(y) + V(x) \geq l, \qquad \text{voor alle } x \in \mathbb{R},$$
$$2\int \log|x-y|^{-1}\,d\mu_V(y) + V(x) = l, \qquad \text{voor alle } x \in \operatorname{supp} \mu_V.$$

voor een zekere l.
Gebruik makend van de Deift-Zhou steilste afdalingsmethode bewijzen we dat

Stelling 7.3.1. Er bestaan constanten α_{2m} en β_m, $m = 1, 2, \dots$ (afhankelijk van V) zodanig dat $a_{n,n}$ en $b_{n,n}$ de volgende asymptotische reeksen bezitten voor $n \to \infty$:

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \qquad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m}.$$

De eerste coëfficiënt β_1 in de reeks voor $b_{n,n}$ wordt gegeven door

$$\beta_1 = \frac{1}{2\pi(b-a)} \left(\frac{1}{h(b)} - \frac{1}{h(a)} \right)$$

Het opmerkelijke van deze stelling is dat de asymptotische reeks voor de $a_{n,n}$ is uitgedrukt in even machten van $\frac{1}{n}$: Gebruik makend van de Deift-Zhou steilste afdalingsmethode slaagt men er grof gezegd altijd in, gegeven de bereidheid een aanzienlijke hoeveelheid berekeningen te doen, een eindig aantal termen in de asymptotische reeks te bepalen. Waar wij in geslaagd zijn is te laten zien dat alle oneven machten in de asymptotische reeks van $a_{n,n}$ wegvallen.
Het dient verder opgemerkt te worden dat voor polynomiale V dit resultaat al in algemenere zin bewezen was door Bleher en Its (zie [10]). Echter, waar zij extra theorie nodig hebben in de vorm van 'snaar vergelijkingen', verkrijgen wij ons resultaat rechtstreeks uit de Deift-Zhou steilste afdalingsmethode.
Dit hoofdstuk is gebaseerd op het gepubliceerde werk [45].

7.4 In hoofdstuk 5:

In hoofdstuk 5 introduceren we een methode gebaseerd op [53] en [54] waarmee resultaten met betrekking tot het limietgedrag van reproducerende kernen voor analytische gewichten gegeneraliseerd kan worden naar het geval dat het gewicht continu of zelfs maar op een deel van de drager van de gewichtsfunctie continu is.

Een centrale rol is in dit hoofdstuk weg gelegd voor de volgende drie functies:

- Laat $\alpha > -1$, $\beta > -1$, $x_0 \in (-1, 1)$ en

$$\nu_{x_0}(z) = \begin{cases} c^2 \text{ voor Re } z \geq x_0 \\ 1 \text{ voor Re } z < x_0 \end{cases} \tag{7.4.1}$$

  met $c > 0$ en $c \neq 1$.

- Laat
$$w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{7.4.2}$$
waarbij $\alpha > -1$, $\beta > -1$ en $x \in [-1,1]$.
- Laat
$$w_N(x) = H(x)e^{-NV(x)}$$
Hier is V reëel analytisch en
$$\frac{V(x)}{\log(1+x^2)} \to +\infty \text{ voor } x \to \pm\infty \tag{7.4.3}$$
Verder is $H(x)$ een functie die positief en continu is op $\operatorname{supp}\psi_V$ met $\operatorname{supp}\psi_V$ als in hoofdstuk 4.

De resultaten van onze methode zijn als volgt:

Stelling 7.4.1. Laat c_1, $c_2 \in \mathbb{R}_{>0}$. Definieer een positieve, eindige Borel maat $d\mu(x) = h(x)w^{\alpha,\beta}(x)dx$ op $(-1,1)$, waarbij $c_1 \leq h(x) \leq c_2$ voor $x \in [-1,1]$ en continu op een open deelinterval $I \subset (-1,1)$.
Laat $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$. Voor $x \in I$ en u, v in compacte deelverzamelingen van $\mathbb{R}$, geldt dat uniform

$$\lim_{n\to\infty} \frac{1}{n\xi(x)} K_n\left(x + \frac{u}{n\xi(x)}, x + \frac{v}{n\xi(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}.$$

Stelling 7.4.2. Laat $\delta > 0$ and c_1, $c_2 \in \mathbb{R}_{>0}$. Definieer een positieve Borel maat $d\mu(x) = h(x)w^{(\alpha,\beta)}(x)dx$ op $(-1,1)$, waarbij $\alpha, \beta > -1$ en

$$c_1 \leq h(x) \leq c_2$$

voor $x \in [-1,1]$ en laat $h(x)$ continu zijn voor $x \in [1-2\delta, 1] \subset [-1,1]$. Dan geldt voor u, v in compacte deelverzamelingen van $(0,\infty)$ dat uniform

$$\lim_{n\to\infty} \frac{1}{2n^2} K_n\left(1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}\right) = \mathbb{J}_{\alpha}(u,v)$$

Stelling 7.4.3. Laat $I \subset [-1,1]$ een open deelinterval van $(-1,1)$, $x_0 \in I$ en $c_1, c_2 \in \mathbb{R}_{>0}$. Definieer een positieve Borel maat $d\mu(x) = h(x)\nu_{x_0}(x)w^{(\alpha,\beta)}(x)dx$ op $(-1,1)$ waarbij $\alpha, \beta > -1$, $x_0 \in (-1,1)$, h continu is op I en

$$c_2 \leq h(x) \leq c_2$$

voor $x \in [-1,1]$. Dan geldt voor $x \in I$ en u, v in compacte deelverzamelingen van $\mathbb{R}$ dat uniform

$$\lim_{n\to\infty} \frac{1}{n\xi(x_0)} K_n\left(x_0 + \frac{u}{n\xi(x_0)}, x_0 + \frac{v}{n\xi(x_0)}\right) = \mathbb{K}_c^{CHF}(u,v)$$

waarbij $\xi(x) = \frac{1}{\pi\sqrt{1-x^2}}$

Stelling 7.4.4. Laat $K_{n,N}(x,y)$ de genormaliseerde reproducerende kern met betrekking tot een gewichtsfunctie $w_N(x) = H(x)e^{-NV(x)}$ zijn.

(a) Voor $\psi_V(x) > 0$ en voor u, v in compacte deelverzamelingen van $\mathbb{R}$ geldt dat uniform

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x)} K_{n,n}\left(x + \frac{u}{n\psi_V(x)}, x + \frac{v}{n\psi_V(x)}\right) = \frac{\sin(\pi(u-v))}{\pi(u-v)}$$

(b) Voor b een rechtereindpunt van $\{x : \psi_V > 0\}$ en u, v in compacte deelverzamelingen van $\mathbb{R}$ geldt dat uniform:

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b + \frac{u}{(cn)^{2/3}}, b + \frac{v}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

waarbij Ai de Airy functie is.

Stelling 7.4.1 en Stelling 7.4.2 waren al bewezen voor een algemener geval door Lubinsky in respectievelijk [54] en [53]. Deel (a) van Stelling 7.4.4 was al bewezen voor een algemener geval door Levin en Lubinsky in [51]. Stelling 7.4.3 en deel (b) van Stelling 7.4.4 zijn nieuwe resultaten.

Appendix A

A Short Overview On Special Functions

A.1 Introduction

For convenience sake, this appendix will serve as a quick overview of the special functions studied within this thesis.

A.2 The Airy function

The Airy function Ai is defined through

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_C e^{zt - \frac{t^3}{3}} \, dt$$

where $C = C_1$ (see Figure A.1) and is a solution to the differential equation

$$y''(z) = zy(z)$$

The main results that we will need about $\mathrm{Ai}(z)$ are Theorem A.2.1 and Theorem A.2.2

Theorem A.2.1.

$$\mathrm{Ai}(z) + \omega \mathrm{Ai}(\omega z) + \omega^2 \mathrm{Ai}(\omega^2 z) = 0$$

where $\omega = e^{2\pi i/3}$.

Theorem A.2.2. For $z \to \infty$, $-\pi < \arg z < \pi$, (see [37])

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi} z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(3k + \frac{1}{2}\right)}{9^k (2k)! \Gamma\left(\frac{1}{2}\right)} \frac{1}{z^{\frac{3}{2}k}} \tag{A.2.1}$$

Theorem A.2.1 and Theorem A.2.2 were taken from [37] and [56].

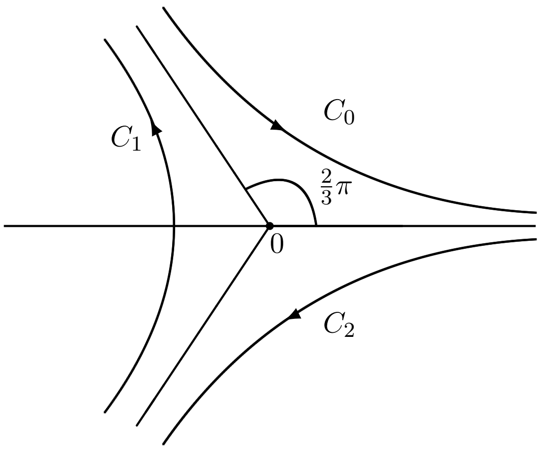

Figure A.1: Sketch of curves for which the integral representation of $\mathrm{Ai}(z)$ is well defined

A.3 Bessel functions

The Bessel function is the solution

$$J_{\pm\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{\pm\nu+2m}}{m!\Gamma(\pm\nu + m + 1)} \tag{A.3.1}$$

to the differential equation

$$z^2 y''(z) + z y'(z) + \left(z^2 - \nu^2\right) y(z) = 0 \tag{A.3.2}$$

Other solutions to (A.3.2) are the Hankel functions

$$H_\nu^{(1)}(z) = i \frac{e^{-\nu\pi i} J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{A.3.3}$$

and

$$H_\nu^{(2)}(z) = i \frac{J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z)}{\sin(\nu\pi)} \tag{A.3.4}$$

If ν is an integer k, we replace (A.3.3) and (A.3.4) by the limits where ν goes to k.
The main results needed regarding Bessel functions are

Theorem A.3.1. For z approaching infinity,

$$
\begin{aligned}
H_\nu^{(1)}(z) &\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n(1/2+\nu)_n}{n!(2iz)^n}, \\
H_\nu^{(2)}(z) &\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n(1/2+\nu)_n}{n!(-2iz)^n}, \\
J_\nu(z) &\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right) \sum_{n=0}^{\infty} \frac{(-1)^n(1/2-\nu)_{2n}(1/2+\nu)_{2n}}{(2n)!(2z)^{2n}}\right. \\
&\quad \left. + \sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right) \sum_{n=0}^{\infty} \frac{(-1)^n(1/2-\nu)_{2n+1}(1/2+\nu)_{2n+1}}{(2n+1)!(2z)^{2n+1}}\right), \\
J_{-\nu}(z) &\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right) \sum_{n=0}^{\infty} \frac{(-1)^n(1/2-\nu)_{2n}(1/2+\nu)_{2n}}{(2n)!(2z)^{2n}}\right. \\
&\quad \left. + \sin\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right) \sum_{n=0}^{\infty} \frac{(-1)^n(1/2-\nu)_{2n+1}(1/2+\nu)_{2n+1}}{(2n+1)!(2z)^{2n+1}}\right)
\end{aligned}
$$

Theorem A.3.2. For z close to 0

$$
\begin{aligned}
J_\nu(z) &= \mathcal{O}\left(z^\nu\right), \\
H_\nu^{(1)}(z) &= \mathcal{O}\left(z^\nu\right), \\
H_\nu^{(2)}(z) &= \mathcal{O}\left(z^\nu\right)
\end{aligned}
$$

These results were taken from [30] and [66].

A.4 Modified Bessel Functions

Modified Bessel functions are solutions to a variant of the Bessel differential equation, namely

$$z^2y''(z) + zy'(z) - (z^2+\nu^2)y(z) = 0$$

Thus, we have the modified Bessel function of the first kind $I_\nu(z)$, defined to be

$$I_\nu(z) = \begin{cases} e^{-\frac{1}{2}\nu\pi i} J_\nu(ze^{\frac{1}{2}\pi i}) \text{ for } -\pi < \arg z \leq \frac{1}{2}\pi \\ e^{\frac{3}{2}\nu\pi i} J_\nu(ze^{-\frac{3}{2}\pi i}) \text{ for } \frac{1}{2}\pi < \arg z \leq \pi \end{cases} \tag{A.4.1}$$

or

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)} \tag{A.4.2}$$

and the modified Bessel function of the second kind

$$K_\nu(z) = \frac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)} \text{ for } \nu \notin \mathbb{N}$$

For $n \in \mathbb{N}$

$$K_n(z) = \lim_{\nu \to n} \frac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}$$

Theorem A.4.1. For z approaching infinity

$$\begin{aligned}
I_\nu(z) \sim{} & \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n (1/2+\nu)_n}{n!(2z)^n} \\
& + \frac{e^{-z-(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n (1/2-\nu)_n (1/2+\nu)_n}{n!(2z)^n} \\
& \text{for } -\pi < \arg z \leq \frac{1}{2}\pi \\
I_\nu(z) \sim{} & \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n (1/2+\nu)_n}{n!(2z)^n} \\
& + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n (1/2-\nu)_n (1/2+\nu)_n}{n!(2z)^n} \\
& \text{for } \frac{1}{2}\pi < \arg z \leq \pi \\
K_\nu(z) \sim{} & \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{n=0}^{\infty} \frac{(1/2-\nu)_n (1/2+\nu)_n}{n!(-2z)^n}
\end{aligned}$$

Proof. This is just a matter of combining the asymptotic formula we already had for $J_\nu(z)$ with the definitions of $I_\nu(z)$ and $K_\nu(z)$. □

And after that, we of course want to know the local behaviour around 0 for the modified Bessel functions as well:

Theorem A.4.2. For z close to zero

$$\begin{aligned}
I_\nu(z) &= \mathcal{O}(z^\nu) \\
K_\nu(z) &= \mathcal{O}(z^\nu) \text{ for } \nu \neq 0 \\
K_0(z) &= \mathcal{O}(\log|z|)
\end{aligned}$$

Proof. As before, the proof is a short exercise in combining the asymptotic behaviour, for z going to zero this time, of the Bessel functions with the definitions of the modified Bessel functions. □

A.5 Alternate proof to Theorem 3.1.1

In this section, we will give an alternate, straightforward proof for Theorem 3.1.1. Recall that the *Bessel kernel* is defined as

$$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}$$

where J_α is the Bessel function.
We will set out to prove that for $s > 0$, for all $x, y \in \mathbb{R}$ and $\alpha > -1$,

$$2\pi s \mathbb{J}_\alpha\left(s^2 + 2\pi x s, s^2 + 2\pi y s\right) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \mathcal{O}\left(\frac{1}{s}\right) \tag{A.5.1}$$

as $s \to \infty$.
Note that for z approaching $+\infty$, we have that by Theorem A.3.1

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right)\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{A.5.2}$$

Using (3.5.1) then gives

$$J'_\alpha(z) = -\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right)\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{A.5.3}$$

for $z \to +\infty$.
Using (A.5.2) and (A.5.3) we find that for $x,\, y \to +\infty$

$$\begin{aligned}\mathbb{J}_\alpha\left(x,y\right) = \frac{1}{\pi(x-y)} &\left(\frac{x}{y}\right)^{\frac{1}{4}} \left(\cos\left(\sqrt{y} - \frac{1}{2}\pi\alpha - \frac{1}{4}\pi\right)\sin\left(\sqrt{x} - \frac{1}{2}\pi\alpha - \frac{1}{4}\pi\right)\right. \\ &\left. - \left(\frac{y}{x}\right)^{\frac{1}{4}} \cos\left(\sqrt{x} - \frac{1}{2}\pi\alpha - \frac{1}{4}\pi\right)\sin\left(\sqrt{y} - \frac{1}{2}\pi\alpha - \frac{1}{4}\pi\right)\right) \\ &\cdot \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{y}}\right)\right)\end{aligned} \tag{A.5.4}$$

Note that

$$\left(\frac{s^2 + 2\pi x s}{s^2 + 2\pi y s}\right)^{\frac{1}{4}} = 1 + \mathcal{O}\left(\frac{x-y}{s}\right) \text{ for } s \to \infty \tag{A.5.5}$$

Replacing x by $s^2 + 2\pi x s$ and y by $s^2 + 2\pi y s$ in (A.5.4) and using (A.5.5) and the symmetry of (A.5.4), we get that for $s \to \infty$

$$\mathbb{J}_\alpha\left(s^2 + 2\pi x s, s^2 + 2\pi y s\right) = \frac{\sin\left(\sqrt{s^2 + 2\pi x s} - \sqrt{s^2 + 2\pi y s}\right)}{2\pi^2 s (x-y)} + \mathcal{O}\left(\frac{1}{s^2}\right) \tag{A.5.6}$$

Note that for $s \to \infty$

$$\sqrt{s^2 + 2\pi x s} = s\sqrt{1 + \frac{2\pi x}{s}} = s\left(1 + \frac{\pi x}{s} + \mathcal{O}\left(\frac{x^2}{s^2}\right)\right)$$

This means that for $s \to \infty$

$$\sqrt{s^2 + 2\pi x s} - \sqrt{s^2 + 2\pi y s} = \pi(x-y) + \mathcal{O}\left(\frac{x-y}{s}\right) \tag{A.5.7}$$

Inserting (A.5.7) into (A.5.6) then gives

$$\mathbb{J}_\alpha\left(s^2 + 2\pi x s, s^2 + 2\pi y s\right) = \frac{\sin\left(\pi(x-y)\right)}{2\pi^2 s(x-y)} + \mathcal{O}\left(\frac{1}{s^2}\right) \tag{A.5.8}$$

Multiplying both sides of (A.5.8) with $2\pi s$ then gives the desired result.

A.6 Confluent Hypergeometric Functions

We will consider the confluent hypergeometric functions $\phi(a, c; z)$ and $\psi(a, c; z)$, solutions to the differential equation

$$zy''(z) + (c - z)y'(z) - ay(z) = 0$$

as defined in [30] through

$$\phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}dt \tag{A.6.1}$$

and

$$\psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\alpha}} t^{a-1}(1+t)^{c-a-1}e^{-zt}dt$$

where $-\frac{1}{2}\pi < \arg z - \alpha < \frac{1}{2}\pi$.
Note that these expressions are not valid for all a and c, but through analytic continuation and contour deformation one can construct integral expressions that demand no restrictions on the parameters (see [30]).
However, for deducing the desired identities, the aforementioned expressions will suffice, where [30] can be invoked if need be.

A.6.1 Basic identities

When working with confluent hypergeometric functions, specifically in this thesis, there are a few basic identities one can't do without. They will be stated in this section with sketches of their proofs, in the sense that certain properties on the parameters are implicitly imposed. However, generalising past these assumptions is a simple matter of using an analytic continuation argument as was done before for the Bessel functions.

Lemma A.6.1. $\phi(a,c;z) = \frac{e^{\mp\pi i(c-a)}\Gamma(c)}{\Gamma(a)} e^z \psi(a,c;e^{\mp\pi i}z) + \frac{e^{\pm\pi i a}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;z)$

Proof. First, let's assume z to be real and positive. Then

$$\psi(a,c;e^{\pm\pi i}z) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{\mp\pi i}} t^{a-1}(1+t)^{c-a-1} e^{-e^{\pm\pi i}zt} dt \tag{A.6.2}$$

Through a change of variable, (A.6.2) can be rewritten as

$$\begin{aligned}
\psi(a,c;e^{\pm\pi i}z) &= \frac{1}{\Gamma(a)} \int_0^{\infty} (te^{\mp\pi i})^{a-1}(1+te^{\mp\pi i})^{c-a-1} e^{-zt} e^{\mp\pi i} dt \\
&= \frac{e^{\mp\pi i a}}{\Gamma(a)} \int_0^{\infty} t^{a-1}(1+te^{\mp\pi i})^{c-a-1} e^{-zt} dt \\
&= \frac{e^{\mp\pi i a}}{\Gamma(a)} \int_0^{1} t^{a-1}(1+te^{\mp\pi i})^{c-a-1} e^{-zt} dt \\
&+ \frac{e^{\mp\pi i a}}{\Gamma(a)} \int_1^{\infty} t^{a-1}(1+te^{\mp\pi i})^{c-a-1} e^{-zt} dt
\end{aligned}$$

Using that

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^{1} t^{a-1}(1+te^{\mp\pi i})^{c-a-1} e^{-zt} dt = \phi(a,c;-z)$$

and taking $e^{\mp\pi i}$ out of the second integral, we get

$$\begin{aligned}
\psi(a,c;e^{\pm\pi i}z) = {} & \frac{e^{\mp\pi i a}\Gamma(c-a)}{\Gamma(c)} \phi(a,c;-z) \\
& - \frac{e^{\mp\pi i c}}{\Gamma(a)} \int_1^{\infty} t^{a-1}(t-1)^{c-a-1} e^{-zt} dt
\end{aligned}$$

A change of variable in the second integral gives

$$\begin{aligned}
\psi(a,c;e^{\pm\pi i}z) &= \frac{e^{\mp\pi i a}\Gamma(c-a)}{\Gamma(c)} \phi(a,c;-z) \\
&- \frac{e^{\mp\pi i c}}{\Gamma(a)} \int_0^{\infty} (t+1)^{a-1} t^{c-a-1} e^{-z(t+1)} dt \\
&= \frac{e^{\mp\pi i a}\Gamma(c-a)}{\Gamma(c)} \phi(a,c;-z) - \frac{e^{\mp\pi i c}\Gamma(c-a)}{\Gamma(a)} e^{-z}\psi(c-a,c;z)
\end{aligned}$$

Dividing both sides by $\frac{e^{\mp\pi i a}\Gamma(c-a)}{\Gamma(c)}$ then gives

$$\frac{\Gamma(c)}{\Gamma(c-a)}e^{\pm\pi i a}\psi(a,c;e^{\pm\pi i}z) = \phi(a,c;-z) - \frac{e^{\mp\pi i(c-a)}\Gamma(c)}{\Gamma(a)}e^{-z}\psi(c-a,c;z)$$

which proves our theorem for $z > 0$. Note that by analyticity of ϕ, the same result holds for all z. $\square$

Lemma A.6.2.

$$\phi(a,c;z) = \phi(c-a,c;-z)e^{z}$$

Proof. Note that if $\phi(a,c;z)$ is a solution to

$$zw'' + (c-z)w' - aw = 0 \tag{A.6.3}$$

then inserting $\phi(c-a,c;-z)e^{z}$ into (A.6.3) proves it to be a solution of (A.6.3) as well (see [2]). Using that every solution to (A.6.3) must be a linear combination of $\psi(a,c;z)$ and $\phi(a,c;z)$ and taking the behaviour of $\phi(c-a,c;-z)e^{z}$ around $z=0$ into account, we find that in fact $\phi(a,c;z) = \phi(c-a,c;-z)e^{z}$ $\square$

Lemma A.6.3.

$$c\phi(a,c;z) = c\phi(a+1,c;z) - z\phi(a+1,c+1;z)$$

Proof.

$$\begin{aligned} c\phi(a+1,c;z) - z\phi(a+1,c+1;z) &= c\sum_{n=0}^{\infty}\frac{(a+1)_n z^n}{(c)_n n!} - z\sum_{n=0}^{\infty}\frac{(a+1)_n z^n}{(c+1)_n n!} \\ &= c + c\sum_{n=1}^{\infty}\frac{(a+1)_n z^n}{(c)_n n!} - \sum_{n=0}^{\infty}\frac{(a+1)_n z^{n+1}}{(c+1)_n n!} \end{aligned} \tag{A.6.4}$$

Rewriting the first infinite sum on the right hand side of (A.6.4) so that the sum starts at $n=0$ instead of $n=1$ gives

$$\begin{aligned} c\phi(a+1,c;z) - z\phi(a+1,c+1;z) &= c + c\sum_{n=0}^{\infty}\frac{(a+1)_{n+1} z^{n+1}}{(c)_{n+1}(n+1)!} - \sum_{n=0}^{\infty}\frac{(a+1)_n z^{n+1}}{(c+1)_n n!} \\ &= c + \sum_{n=0}^{\infty}\frac{(a+1)_{n+1} z^{n+1}}{(c+1)_n (n+1)!} - \sum_{n=0}^{\infty}\frac{(a+1)_n z^{n+1}}{(c+1)_n n!} \end{aligned} \tag{A.6.5}$$

Rewriting the two infinite sums on the right hand side of (A.6.5) as one sum then leads to

$$
\begin{aligned}
c\phi(a+1,c;z) - z\phi(a+1,c+1;z) &= c + \sum_{n=0}^{\infty} \frac{(a+1)_n z^{n+1}}{(c+1)_n} \left(\frac{a+1+n}{(n+1)!} - \frac{1}{n!} \right) \\
&= c + \sum_{n=0}^{\infty} \frac{(a+1)_n z^{n+1}}{(c+1)_n} \left(\frac{a}{(n+1)!} \right) \\
&= c + \sum_{n=0}^{\infty} \frac{c(a)_{n+1} z^{n+1}}{c(c+1)_n (n+1)!} \\
&= c \left(1 + \sum_{n=1}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \right) = c\phi(a,c;z)
\end{aligned}
$$

□

Lemma A.6.4.

$$
\begin{aligned}
\frac{d^n}{dz^n}\phi(a,c;z) &= \frac{(a)_n}{(c)_n}\phi(a+n,c+n;z) \\
\frac{d^n}{dz^n}\psi(a,c;z) &= (-1)^n (a)_n \psi(a+n,c+n;z)
\end{aligned}
$$

Proof. Starting with ϕ, we find that

$$
\begin{aligned}
\frac{d^n}{dz^n}\phi(a,c;z) &= \frac{d^n}{dz^n} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt} dt \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a+n-1}(1-t)^{c-a-1} e^{zt} dt \\
&= \frac{(a)_n}{(c)_n} \frac{(c)_n}{(a)_n} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a+n-1}(1-t)^{c+n-a-n-1} e^{zt} dt \\
&= \frac{(a)_n}{(c)_n} \frac{\Gamma(c+n)}{\Gamma(a+n)\Gamma(c+n-(a+n))} \int_0^1 t^{a+n-1}(1-t)^{c+n-(a+n)-1} e^{zt} dt \\
&= \frac{(a)_n}{(c)_n}\phi(a+n,c+n;z)
\end{aligned}
$$

A similar calculation proves the second equality. □

A.6.2 Asymptotics of $\phi(a, c; z)$ and $\psi(a, c; z)$.

We will discuss the asymptotics of $\phi(a, c; z)$ and $\psi(a, c; z)$ for z going to zero and for z approaching infinity.

Theorem A.6.5.
- $$\phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!(c)_n}$$

  where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$

- For $z \to \infty$

  $$\phi(a, c; z) = \frac{e^{\pm a\pi i} z^{-a}\Gamma(c)}{\Gamma(c-a)} \left(\sum_{k=0}^{n} \frac{(1+a-c)_k (a)_k}{k! z^k} + \mathcal{O}\left(\frac{1}{z^{n+1}}\right)\right)$$
  $$+ \frac{z^{a-c}\Gamma(c)}{\Gamma(a)} e^z \left(\sum_{k=0}^{m} \frac{(1-a)_k (c-a)_k}{k! z^k} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right)\right)$$

  where we take the upper sign if $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ and we take the lower sign if $-\frac{3}{2}\pi < \arg z < -\frac{1}{2}\pi$.

- For $z \to 0$

  $$\psi(a, c; z) = \begin{cases} \mathcal{O}(z^{1-c}) \text{ for } \operatorname{Re} c > 1 \\ \mathcal{O}(\log z) \text{ for } c = 1 \end{cases}$$

- For $z \to \infty$

  $$\psi(a, c; z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (1+a+c)_n}{n! z^n}$$

Proof. To start with ϕ:

$$\phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt} dt$$

Expanding e^{zt} as a power series and noting that summation and integration in this case may be interchanged, gives

$$\begin{aligned}\phi(a,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_0^1 t^{a+n-1}(1-t)^{c-a-1}dt\\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{z^n}{n!}B(a+n,c-a)\\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(a+n+c-a)}\\ &= \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!(c)_n}\end{aligned}$$

which proves the first part of our lemma.
Slightly trickier is obtaining the local asymptotic behaviour of ψ. Our plan of attack will be to deduce local characteristics from the jump of ψ on the real axis: From Lemma A.6.1 we learn that

$$\phi(a,c;z) = \frac{e^{\mp\pi i(c-a)}\Gamma(c)}{\Gamma(a)}e^z\psi(a,c;e^{\mp\pi i}z) + \frac{e^{\pm\pi ia}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;z)$$

which can be rewritten as

$$e^{\pm\pi i(c-a)}\phi(a,c;z) - \frac{e^{\pm\pi ic}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}e^z\psi(a,c;e^{\mp\pi i}z) \qquad \text{(A.6.6)}$$

from which, in turn, one can obtain by replacing z by $e^{\pm2\pi i}z$, that

$$e^{\pm\pi i(c-a)}\phi(a,c;e^{\pm2\pi i}z) - \frac{e^{\pm\pi ic}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;e^{\pm2\pi i}z) = \frac{\Gamma(c)}{\Gamma(a)}e^z\psi(a,c;e^{\pm\pi i}z)$$

or, as $\phi(a,c;z)$ is analytic,

$$e^{\pm\pi i(c-a)}\phi(a,c;z) - \frac{e^{\pm\pi ic}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;e^{\pm2\pi i}z) = \frac{\Gamma(c)}{\Gamma(a)}e^z\psi(a,c;e^{\pm\pi i}z) \qquad \text{(A.6.7)}$$

Combining (A.6.6) and(A.6.7) in order to rid ourselves of $\frac{\Gamma(c)}{\Gamma(a)}e^z\psi(a,c;e^{\pm\pi i}z)$, we get

$$\begin{aligned}&e^{\mp\pi i(c-a)}\phi(a,c;z) - \frac{e^{\mp\pi ic}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;z)\\ &\qquad = e^{\pm\pi i(c-a)}\phi(a,c;z) - \frac{e^{\pm\pi ic}\Gamma(c)}{\Gamma(c-a)}\psi(a,c;e^{\pm2\pi i}z)\end{aligned}$$

from which follows that, using that $\frac{e^{ix}-e^{-ix}}{2i} = \sin x$,

$$\psi(a,c;z) - e^{\pm 2\pi i c}\psi(a,c;e^{\pm 2\pi i}z) = \mp 2i \sin(\pi(c-a))e^{\pm\pi i c}\frac{\Gamma(c-a)}{\Gamma(c)}\phi(a,c;z)$$

Particularly,

$$\psi(a,c;z) - e^{2\pi i c}\psi(a,c;e^{2\pi i}z) = -2i \sin(\pi(c-a))e^{\pi i c}\frac{\Gamma(c-a)}{\Gamma(c)}\phi(a,c;z)$$

Remember that our goal was to deduce behaviour around $z = 0$ of ψ. To that end, define a function $g(z)$ through

$$g(z) = \frac{-\psi(a,c;z)e^{-\pi i c}\Gamma(c)}{\Gamma(c-a)\phi(a,c;z)}$$

Then

$$g(z) - e^{2\pi i c}g(e^{2\pi i}z) = 1$$

Clearly, for $c \in \mathbb{Z}$, g behaves like a logarithmic function possibly combined with a Laurent series for z close to 0, as writing $h(z) = g(z) + \frac{1}{2\pi i}\log z$ gives

$$h(z) - h(e^{2\pi i}z) = g(z) + \frac{1}{2\pi i}\log z - g(e^{2\pi i}z) - \frac{1}{2\pi i}\log z - 1 = 1 - 1 = 0$$

revealing h to be analytic away from zero and thus g to behave like a logarithmic function, possibly combined with a Laurent series, around $z = 0$. For $c \notin \mathbb{Z}$, define

$$q(z) = z^b g(z) + r(z)$$

where b is some constant and r is a function that will be defined later. Then

$$\begin{aligned} q(z) - q(e^{2\pi i}z) &= z^b(g(z) - e^{2\pi i b}g(e^{2\pi i}z)) + r(z) - r(e^{2\pi i}z) \\ &= z^b \cdot 1 + r(z) - r(e^{2\pi i}z) \end{aligned}$$

for $b = c(\mod 1)$. Choosing $r(z) = \frac{z^b}{e^{2\pi i b}-1}$ will then lead to

$$q(z) - q(e^{2\pi i}z) = 0$$

showing q to be analytic away from zero and g to behave as $a_b z^{-b} + L(z)$, where $b = c(\text{mod } 1)$ and a_b is some Laurent series. As $g(z)\phi(a,c;z)$ is equal to $\psi(a,c;z)$ up to a constant, the same holds for ψ, as $\phi(a,c;z) = 1 + \mathcal{O}(z)$ for $z \to 0$ and does not affect the local behaviour of g. Obviously, we want to know exactly what b to choose. Thus, recall that ψ is a solution to the confluent hypergeometric differential equation

$$zy''(z) + (c - z)y'(z) - ay(z) = 0$$

For a solution y to behave like a power function z^b around $z = 0$, it is likely that our choice of b will be severely restricted. Thus, writing $y(z) = z^b + H(z)$, where $H(z)$ is of higher order than z^{-b} and inserting y into the differential equation, gives

$$z(-b(-b-1)z^{-b-2} + H''(z)) + (c - z)(-bz^{-b-1} + H'(z)) - a(z^{-b} + H(z)) = 0$$

and consequently

$$-b(-b + c - 1)z^{-b-1} + R(z) = 0$$

where $R(z) = zH''(z) + bz^{-b} - zH'(z) - az^{-b} - aH(z)$ which is of higher order than z^{-b-1}. Hence, $-b(-b+c-1)$ must be equal to zero, so for $c \notin \mathbb{Z}$ either $b = 0$, or $b = c - 1$, . For $c \in \mathbb{Z}$, define $y(z) = \log z + H(z)$, where H is in this case of higher order than $\log z$. Then inserting y into the confluent hypergeometric equation gives

$$z\left(\frac{-1}{z^2} + H''(z)\right) + (c - z)\left(\frac{1}{z} + H'(z)\right) - a(\log z + H(z)) = 0$$

or

$$(c - 1)\frac{1}{z} + R(z) = 0$$

where this time $R(z) = zH''(z) - 1 + (c - z)H'(z) - a(\log z + H(z))$ and of higher order than $\frac{1}{z}$. Thus, $c - 1 = 0$, meaning $c = 1$, proving that only for $c = 1$ a solution can show logarithmic behaviour. For all other entire c, the leading order terms must then, by the previous argument, behave like power functions of the type z^{1-c}. As such, for $c = 1$, $\psi(a, c; z) = \mathcal{O}(\log z)$ and for $c \neq 1$, $\psi(a, c; z) = \mathcal{O}(z^{1-c})$, thereby completing the proof for the local behaviour.

For $z \to \infty$, things are far simpler: Again, starting with $\phi(a, c; z)$:

For Re $z < 0$

$$
\begin{aligned}
\phi(a,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt} dt \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^z \left(\frac{t}{z}\right)^{a-1} \left(1-\frac{t}{z}\right)^{c-a-1} e^t (z^{-1}) dt \\
&= \frac{z^{-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^z t^{a-1} \left(1-\frac{t}{z}\right)^{c-a-1} e^t dt \\
&= \frac{z^{-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{n} \int_0^z t^{a-1} \frac{(1+a-c)_k}{k!} \left(\frac{t}{z}\right)^k e^t dt + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \\
&= \frac{z^{-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{n} \frac{(1+a-c)_k}{k! z^k} \int_0^z t^{k+a-1} e^t dt + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \\
&= \frac{z^{-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{n} \frac{(1+a-c)_k}{k! z^k} \int_0^{-z} (e^{\pm\pi i} t)^{k+a-1} e^{-t} (-1) dt + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \\
&= \frac{z^{-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{n} \frac{(1+a-c)_k}{k!(-z)^k} e^{\pm(k+a)\pi i} \int_0^{-z} t^{k+a-1} e^{-t} dt + \mathcal{O}\left(\frac{1}{z^{n+1}}\right)
\end{aligned}
$$

Note that

$$\int_0^{-z} t^{k+a-1} e^{-t} dt$$

is exponentially decreasing for $|z|$ increasing, so the integral is exponentially close to $\Gamma(k+a)$, meaning that

$$
\begin{aligned}
\phi(a,c;z) =& \frac{z^{-a}\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \sum_{k=0}^{n} \frac{(1+a-c)_k}{k!(-z)^k} e^{\pm(k+a)\pi i} \Gamma(k+a) + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \\
=& \frac{e^{\pm(k+a)\pi i} z^{-a}\Gamma(c)}{\Gamma(c-a)} \sum_{k=0}^{n} \frac{(1+a-c)_k (a)_k}{k!(-z)^k} + \mathcal{O}\left(\frac{1}{z^{n+1}}\right)
\end{aligned}
$$

which completes the proof for Re $z < 0$.
For Re $z > 0$, we recall from Lemma A.6.2 that

$$\phi(a,c;z) = \phi(c-a,c;-z)e^z$$

Repeating the aforementioned procedure for $\phi(c-a,c;-z)$, then gives exactly

$$\phi(a,c;z) = \frac{z^{a-c}\Gamma(c)}{\Gamma(a)} e^z \left(\sum_{k=0}^{m} \frac{(1-a)_k(c-a)_k}{k!z^k} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right) \right)$$

And as, asymptotically speaking the respective expressions for Re $z < 0$ and Re $z > 0$ blot each other out, (when Re $z < 0$, the expression found for Re $z > 0$ is exponentially decreasing and when Re $z > 0$, the expression for Re $z < 0$ grows nowhere nearly as fast as the exponentially increasing expression for Re $z > 0$), so it is allowed to add the two, leading to

$$\begin{aligned}\phi(a,c;z) &= \frac{e^{\pm a\pi i} z^{-a}\Gamma(c)}{\Gamma(c-a)} \left(\sum_{k=0}^{n} \frac{(1+a-c)_k(a)_k}{k!z^k} + \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \right) \\ &\quad + \frac{z^{a-c}\Gamma(c)}{\Gamma(a)} e^z \left(\sum_{k=0}^{m} \frac{(1-a)_k(c-a)_k}{k!z^k} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right) \right)\end{aligned}$$

Finally, we want the asymptotic behaviour for $z \to \infty$ of $\psi(a,c;z)$:

$$\begin{aligned}\psi(a,c;z) &= \frac{1}{\Gamma(a)} \int_0^{e^{i\alpha}\infty} t^{a-1}(1+t)^{c-a-1} e^{-zt} dt \\ &= \frac{1}{\Gamma(a)} \int_0^{\infty} \left(\frac{t}{z}\right)^{a-1} \left(1+\frac{t}{z}\right)^{c-a-1} e^{-t} dt \\ &= \frac{z^{-a}}{\Gamma(a)} \int_0^{\infty} t^{a-1} \left(1+\frac{t}{z}\right)^{c-a-1} e^{-t} dt \\ &\sim \frac{z^{-a}}{\Gamma(a)} \sum_{n=0}^{\infty} \int_0^{\infty} t^{n+a-1} \frac{(-1)^n(1+a-c)_n}{n!z^n} e^{-t} dt \\ &= \frac{z^{-a}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(-1)^n(1+a-c)_n}{n!z^n} \Gamma(a+n) \\ &= z^{-a} \sum_{n=0}^{\infty} \frac{(-1)^n(1+a-c)_n(a)_n}{n!z^n}\end{aligned}$$

which finalises our proof. □

Appendix B

Random matrices and reproducing kernels

This short chapter serves as a quick summary on random matrices, as to further the understanding of the limiting kernels discussed in chapter 3. Random matrices were first introduced into statistics in 1928 by Wishart (see [67]) and have had a profound impact on various fields of mathematics and physics alike. For a more thorough overview on the subject, please see [35] and the standard references [19] and [55].
In this chapter we will start by following [3], [4] and [47] in that we focus on a unitary random matrix ensemble, meaning the space of $n \times n$ Hermitian matrices M, on which we define a probability distribution

$$\begin{aligned} P^{(n)}(M)dM &= \frac{1}{Z_n}|\det M|^{2\alpha} e^{-n\mathrm{tr}V(M)} dM \\ &= \frac{1}{Z_n}|\det M|^{2\alpha} e^{-n\mathrm{tr}V(M)} \prod_{i=1}^{n} dM_{ii} \prod_{i<j} (dM_{ij}^R dM_{ij}^I),\ \alpha > -\frac{1}{2} \end{aligned} \tag{B.0.1}$$

where $M_{kj} = M_{kj}^R + iM_{kj}^I$ denotes the sum of the real and imaginary parts of the matrix entry M_{kj} and Z_n is a normalising constant.
Furthermore, V is a real analytic function for which

$$\frac{V(x)}{\log(1+x^2)} \to \infty \text{ as } x \to \pm\infty$$

Equation B.0.1 gives rise to a probability density function on the eigenvalues $x_1,...,x_n$ of M

$$\widehat{P}^{(n)}(x)d^n x = \frac{1}{\widehat{Z}_n} \prod_{j=1}^{n} w_n(x_j) \prod_{i<j} |x_i - x_j|^2 \tag{B.0.2}$$

(see [19] and [55]) where $\widehat{Z}_n$ is a normalisation constant and

$$w_n(x) = |x|^{2\alpha} e^{-nV(x)},\ \alpha > -\frac{1}{2},\ x \in \mathbb{R}$$

Of particular interest are the limiting mean density ψ_V of the eigenvalues and the so-called $m-$point correlation kernel

$$\mathcal{R}_{n,m}(y_1,\dots,y_m) = \frac{n!}{(n-m)!} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-m} \widehat{P}^{(n)}(y_1,\dots,y_m,x_{m+1},\dots,x_n) \prod_{i=m+1}^{n} dx_i \tag{B.0.3}$$

where $1 \leq m \leq n-1$ (see [19] and [55]).
It was shown in [55] that

$$\mathcal{R}_{n,m}(y_1,\dots,y_m) = \det\left(K_{n,n}(y_i,y_j)\right)_{1\leq i,j\leq m}$$

where

$$K_{n,n}(x,y) = \sum_{i=0}^{n-1} p_{i,n}(x)p_{i,n}(y)w_n(x)^{\frac{1}{2}}w_n(y)^{\frac{1}{2}} \tag{B.0.4}$$

where $p_{i,n}$ is the ith orthonormal polynomial with respect to the weight function w_n . We call $K_{n,n}$ the *normalised reproducing kernel.*
Furthermore, it can be shown that

$$\psi_V(x) = \lim_{n\to\infty} \frac{1}{n} K_{n,n}(x,x)$$

So in order to find the general behaviour of $\mathcal{R}_{n,m}$ and ψ_V, we first need to find the general behaviour of $K_{n,n}$. And this is where the limiting kernels of chapter 3 come in:

- For $\alpha = 0$ and points x^* for which $\psi_V(x*) > 0$, Deift et al. (see[19], [22], [23]) proved that for u, $v \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x^*)} K_{n,n}\left(x^* + \frac{u}{n\psi_V(x^*)}, x^* + \frac{v}{n\psi_V(x^*)}\right) = \frac{\sin\pi(u-v)}{\pi(u-v)}$$

  where we recognise the *sine kernel* from chapter 3. Furthermore, for b a right edge point of $\{x : \psi_V(x) > 0\}$, ψ_V vanishing like a square root around $x = b$ and $u, v \in \mathbb{R}$, Deift et al. proved:

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(b + \frac{u}{(cn)^{2/3}}, b + \frac{v}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{u-v}$$

  where Ai is the Airy function and we refer to the limit kernel as the *Airy kernel.*

- For $\alpha > -\frac{1}{2}$ and $\psi_V(0) > 0$, it was shown by Kuijlaars and Vanlessen (see [47]) that for u, $v \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n\psi_V(0)} K_{n,n}\left(\frac{u}{n\psi_V(0)}, \frac{v}{n\psi_V(0)}\right) = \mathbb{J}^0_\alpha(u,v)$$

where

$$\mathbb{J}^0_\alpha(x,y) = \pi \left(\frac{|x|}{x}\right)^\alpha \left(\frac{|y|}{y}\right)^\alpha \sqrt{x}\sqrt{y} \frac{J_{\alpha+\frac{1}{2}}(\pi x) J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x) J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)}$$

with $J_{\alpha\pm\frac{1}{2}}$ the Bessel function of order $\alpha \pm \frac{1}{2}$ (see [3], [34] and Remark 1.2 of [47]).

Observe that for x^* moving from an endpoint b of $\operatorname{supp}\psi_V$ or from special points such as 0, the limit behaviour of the normalised reproducing kernel should shift from Airy kernel behaviour or Bessel kernel behaviour to sine kernel behaviour, thus indicating a relation between the Bessel kernel $\mathbb{J}^0_\alpha$, the Airy kernel and the sine kernel. A relation between the Airy kernel and the sine kernel was shown in [42], in chapter 3 we have deduced the relation between the sine kernel and $\mathbb{J}^0_\alpha$.
Let's move on to a different random matrix ensemble, being the modified Jacobi unitary ensemble with a jump, which differs from the previous setting in that we now use fixed weights

$$w(x) = h(x)(1-x)^\alpha(1+x)^\beta \nu_{x_0}(x)$$

where $x \in [-1,1]$, h is positive and real analytic on $[-1,1]$, α, $\beta > -1$, $x_0 \in (-1,1)$ and

$$\nu_{x_0}(x) = \begin{cases} c^2 \text{ for } x \geq x_0 \\ 1 \text{ for } x < x_0 \end{cases}$$

with $c > 0$.
We have seen in chapter 2 how the limit behaviour of the normalised reproducing kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)w(x)^{\frac{1}{2}}w(x)^{\frac{1}{2}}$$

where p_k is the kth orthonormal polynomial with respect to the weight w, is related to the *sine kernel*, the *Bessel kernel* $\mathbb{J}_\alpha$ and the *Confluent Hypergeometric kernel* $\mathbb{K}^{CHF}_c(x,y)$.
The *Bessel kernel* $\mathbb{J}_\alpha$ and the *Confluent Hypergeometric kernel* $\mathbb{K}^{CHF}_c(x,y)$ are defined as

- $$\mathbb{K}_c^{CHF}(x,y) = \frac{\nu_0(x)^{\frac{1}{2}}\nu_0(y)^{\frac{1}{2}}\log c}{\pi i(x-y)(c^2-1)}[G(1+\lambda; 2\pi ix); G(\lambda; 2\pi iy)]$$

  where $\lambda = \frac{i\log c}{\pi}$, $G(a;z) = \phi(a,1;z)e^{-\frac{z}{2}}$, with $\phi(a,c;z)$ the confluent hypergeometric function of the first kind and $[f(x);g(y)] = f(x)g(y) - f(y)g(x)$.

- $$\mathbb{J}_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'(\sqrt{x})}{2(x-y)}$$

  where J_α is the Bessel function.

In the same way that one might suspect a link between the sine kernel, the Airy kernel and $\mathbb{J}_\alpha^0$ for $\alpha > -\frac{1}{2}$, a relation between the sine kernel, the Confluent Hypergeometric kernel and $\mathbb{J}_\alpha$ is unavoidable. Furthermore, comparing the local behaviour to the left of 0 in the previous case with the behaviour to the left of 1 in the current case, judging from the respective weight functions, some sort of quadratic relation should exist between J_α and J_α^0. As it turns out, this is exactly what we use in our proof of Theorem 3.1.3.

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